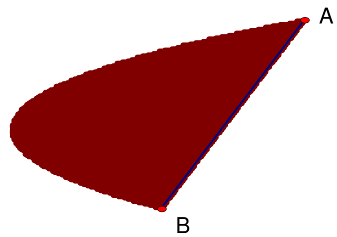
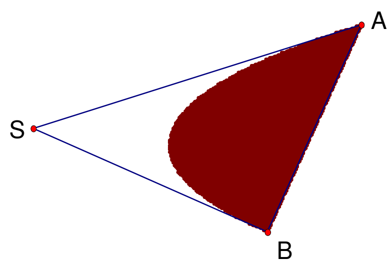
56. Archimedes' Squaring of a Parabola

Determine the area of a parabola section.



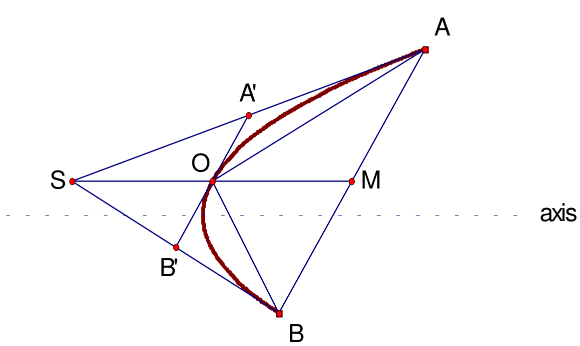
This is one of Archimedes' most remarkable achievements. It dates to about 240 B.C., and is based on properties of Archimedean triangles. An Archimedean triangle is a triangle whose sides consist of two tangents to a parabola and the chord joining the points of tangency:



The chord AB is the base of the triangle. The following result is true about Archimedean triangles.

Theorem.

1. The median to the base of an Archimedean triangle is parallel to the axis of the parabola,



2. the midline parallel to the base is tangent to the parabola at the intersection point O of it and the median to the base. O is the midpoint of segment AB.
3. The area of the "internal triangle" $\triangle AOB$ is half the area of the Archimedean triangle $\triangle ASB$.

4. The area of the "external triangle" $\triangle A'SB'$ is $\frac{1}{4}$ the area of $\triangle ASB$.
5. The area of the "residual" Archimedean triangle $\triangle AOA'$ $\left(\text{and } \triangle BOB'\right)$ is $\frac{1}{8}$ the area of $\triangle ASB$.

We'll return to the proofs later. Here's how Archimedes arrived at his conclusion. Let $\triangle$ be the area of $\triangle ASB$. Then the area of the internal triangle $\triangle AOB$ is $\frac{1}{2}\triangle$. Repeat this computation for $\triangle AOA'$ and $\triangle AOA'$ to get an area (for the internal triangles) of $2\left(\frac{1}{2}\cdot\frac{\triangle}{8}\right) = \frac{1}{2}\cdot 2\cdot\frac{\triangle}{8}$. The next four internal triangles have a combined area of $\frac{1}{2}\cdot 4\cdot\frac{\triangle}{8^2}$, etc. This process "exhausts" the parabola section, and thus its area is

$$\frac{1}{2}\left[\triangle + 2\cdot\frac{\triangle}{8} + 4\cdot\frac{\triangle}{8^2} + 8\cdot\frac{\triangle}{8^3} + \ldots\right] = \frac{\triangle}{2}\left[1 + \frac{1}{4} + \frac{1}{4^2} + \frac{1}{4^3} + \ldots\right] = \frac{2}{3}\triangle,$$

or **the area enclosed by a parabola section is two thirds the area of the corresponding Archimedean triangle**.

Proofs of Theorems 3, 4 and 5 follow from 1 and 2, so it suffices to prove 1 and 2. Dörrie proves 1 and 2 synthetically, i.e., using geometric properties of parabolas. Analytic proofs can also be given. For instance, if the parabola is $y = \frac{1}{4p}x^2$, and $A = \left(a, \frac{1}{4p}a^2\right)$, $B = \left(b, \frac{1}{4p}b^2\right)$, then $M = \left(\frac{a+b}{2}, \frac{a^2+b^2}{8p}\right)$, $S = \left(\frac{a+b}{2}\ \frac{ab}{4p}\right)$ and 1 follows. $A' = \left(\frac{3a+b}{4}, \frac{a(a+b)}{8p}\right)$, $B' = \left(\frac{a+3b}{4}, \frac{b(a+b)}{8p}\right)$ and $O = \left(\frac{a+b}{2}, \frac{1}{4p}\left(\frac{a+b}{2}\right)^2\right)$, and 2 follows.
$\square$

Note 1. With the notation above, assuming that $a < b$, the area of the parabola section by Calculus is $\int_a^b\left(\frac{a+b}{4p}x - \frac{ab}{4p} - \frac{x^2}{4p}\right)dx = \frac{(b-a)^3}{24p}$.

Note 2. The area of the Archimedean triangle $\triangle ASB$ is the absolute value of

$$\frac{1}{2}\begin{vmatrix} a & \frac{1}{4p}a^2 & 1 \\ b & \frac{1}{4p}b^2 & 1 \\ \frac{a+b}{2} & \frac{ab}{4p} & 1 \end{vmatrix} = \frac{1}{16p}(a-b)^3,$$

and we see that the area of the parabola section is two thirds the area of the corresponding Archimedean triangle.

Note 3. Dörrie concludes with another way of writing the area. $4p$ is the **parameter** or length of the **latus rectum** of the parabola, $b - a$ is the section transverse, i.e., the length of the normal projection of the section on the directrix. Then six times the latus rectum (parameter) and the area equals the cube of the section transverse.