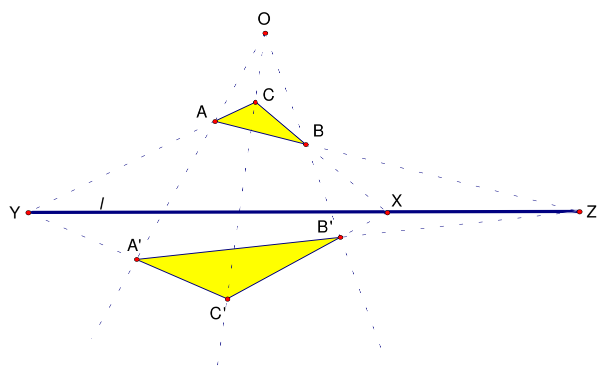
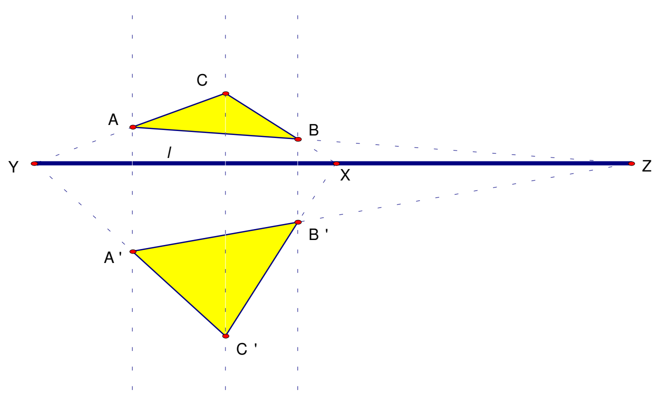
59. Desargues' Theorem.

Two triangles that are perspective from a point are perspective from a line, and converseley, two triangles that are perspective from a line are perspective from a point.

Triangles $\triangle ABC$ and $\triangle A'B'C'$ are perspective from a point O if lines AA', BB' and CC' meet at O. Triangles $\triangle ABC$ and $\triangle A'B'C'$ are perspective from a line l if the points $X = BC \cap B'C', Y = AC \cap A'C'$ and $Z = AB \cap A'B'$ lie on l.



Note 1. This is a theorem in projective geometry, more specifically in the augmented or extended Euclidean plane. Thus the point O might be "at infinity":



Note 2. In the axiomatic development of projective geometry, Desargues' Theorem is often taken as an axiom.

Dörrie begins by providing the reader with a short exposition of the most important concepts and simplest theorems of projective geometry.

Definitions.

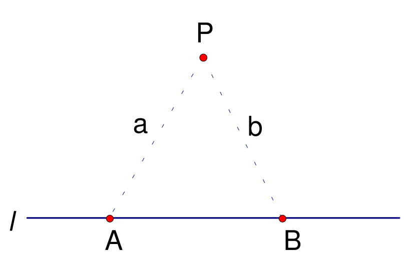
1. A *range or pencil of points* is the set of all points on a line; the line is the base of the range.
2. A *field of points* is the set of all points on a circle or conic section.
3. A *pencil of lines* is the set of all lines through a fixed point P; P is the center of the pencil.
4. A *field of tangents* is the set of all tangent lines to a circle or conic section.
5. The *basic structures* of plane projective geometry are pencils of points, lines and fields of tangents.
6. The *elements* are points, lines and individual tangent lines.
7. The *cross ratio* of
    a. Four points A,B,C,D on a line is the ratio $(A,B,C,D) = \frac{AC}{BC} \div \frac{AD}{BD}$, where directed distances are used,
    b. Four lines a,b,c,d of a pencil of lines is $(a,b,c,d) = \frac{\sin ac}{\sin bc} \div \frac{\sin ad}{\sin bd}$, where ac is the angle between lines a and c, etc.
    c. Four points on a circle or conic section is the cross ratio of the four lines that connect the four points with a fifth point on the circle or conic section. (cf. No. 61),
    d. Four tangents from a field of tangents is the cross ratio of the points of tangency.
8. A *projectivity* is a one-to-one correspondence between elements of two structures in which corresponding elements have the same cross-ratio. Structures R and S are *projective* if there is a projectivity betweenthem, and we write $R \barwedge S$ to indicate this.

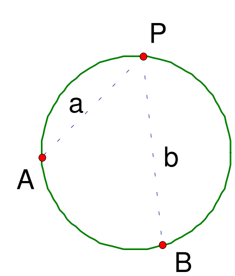
The Fundamental Theorem says that a projectivity is uniquely determined from the correspondence of three elements of one structure with three elements of the other one. (This follows from the property of cross ratios that if $(A,B,C,D) = (A,B,C,D')$, then $D = D'$.)

Note 3. Modern treatments of projective geometry usually begin with *perspectivities* rather than projectivities. A projectivity is a composition of perspectivities.

A perspectivity between

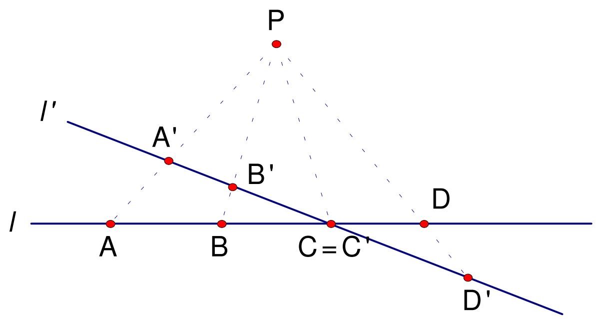
1. A range of points and a pencil of lines is a correspondence in which each point lies on the corresponding line:



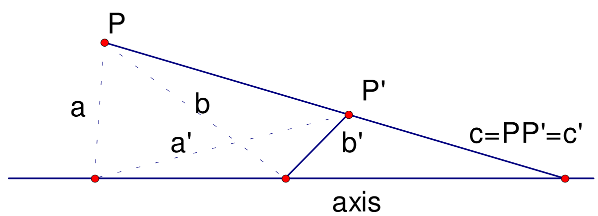


Each line is called the reflection of the corresponding point, and the whole pencil is the reflection of the range of points.

2. Two pencils of points on different lines (or just two lines) is a correspondence in which the lines through corresponding points all pass through a single point, the center of perspectivity:



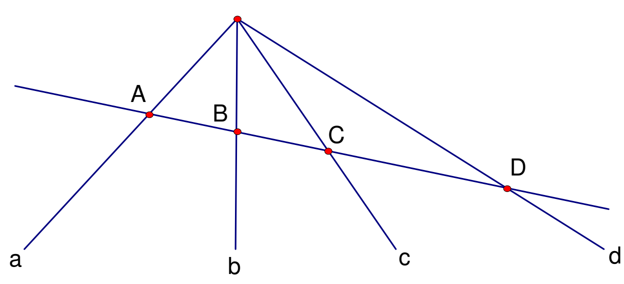
3. Two pencils of lines through different points is a correspondence in which the intersection points of corresponding points lie on a single line, the axis of perspectivity:



$\overline{\overline{\wedge}}$ is used to denote a perspectivity between two structures.

Here are some other well-known theorems from projective geometry:

Pappus' Theorem The cross ratio of four lines of a pencil of lines equals the cross ratio of the four points at which an arbitrary line cuts the lines. (From *Collectiones mathematicae*, Pappus of Alexandria, fourth century A.D.)



(a,b,c,d) = (A,B,C,D)

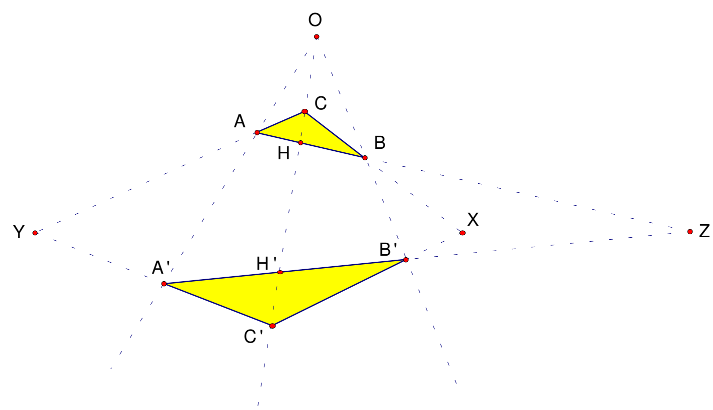
Theorem 3. If in a projectivity between two lines the point of intersection corresponds to itself, the projectivity is a perspectivity.

Theorem 4. If in a projectivity between two pencils of lines the line joining the two centers corresponds to itself, the projectivity is a perpsectivity.

Note 4. Since our main interest is in proving Desargues' Theorem, we will defer the proofs for the time being.

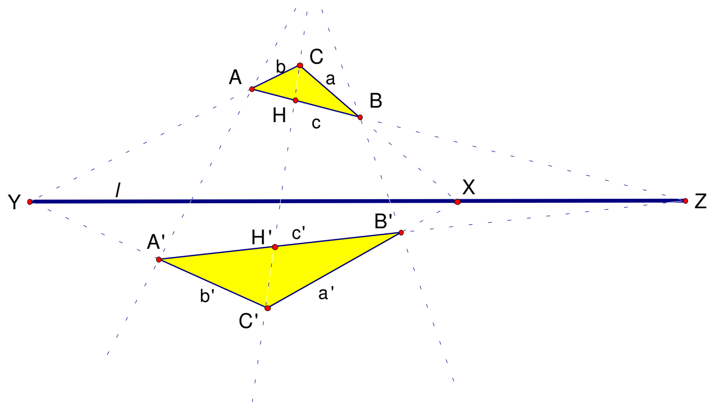
Proof of Desargues' Theorem. Let $\triangle ABC$ with (extended) sides a, b, c opposite the vertices and $\triangle A'B'C'$ with (extended) sides a', b', c' be the corresponding trianges. Let X, Y, Z be the intersection points of a ad a', b and b' and c and c' respectively, and let H and H' be points of intersection of line CC' with (lines) AB and $A'B'$ respectively. (Some point might be points at infinity in the extended Euclidean plane.) There are two parts to the proof.

I. Assume that the lines AA', BB' and CC' meet at point O.



The perspectivity $c \overset{=}{\wedge} c'$ from O takes A, B, H, Z to $A', B', H', Z' = Z$, and by Pappus' Theorem $(A, B, H, Z) = (A', B', H', Z)$. By the Fundamental Theorem, there is a unique projectivity taking lines $CA, CB, CH = CC'$ to $C'A', C'B', C'H' = CC'$. Line CZ maps to some line $C'W$ (with W on line $A'B'$). Then $(A, B, H, Z) = (A', B', H', W)$, but $(A, B, H, Z) = (A', B', H', Z)$ too, and thus $W = Z$. Therefore $CA, CB, CH = CC', CZ$ and $C'A', C'B', C'H' = CC', C'Z$ are corresponding lines of a projectivity of the pencils through C and C'. Since CC' is fixed, the projectivity is a perspectivity, and the points of intersection of corresponding lines lie on a line, i.e. $CA \cap C'A' = Y$, $CB \cap C'B' = X$ and $CZ \cap C'Z = Z$ are collinear.

II. Now assume that the points $X = a \cap a'$, $Y = b \cap b'$ and $Z = c \cap c'$ lie on a line l.

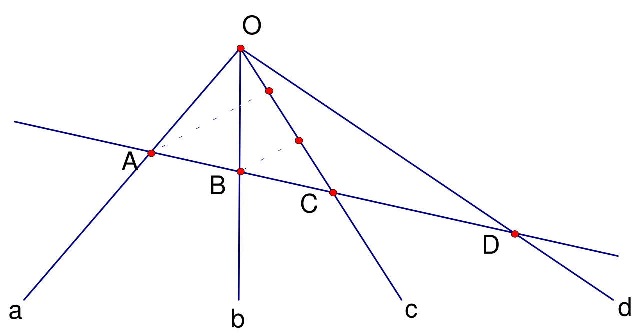


Join the points of line l with C and C' to get a perspective correspondence between the pencils of lines through C and C' in which a, b, CC', CZ correspond to $a', b', C'C, C'Z$. Since $(a, b, CC', CZ) = (a', b', C'C, C'Z)$, Pappus' Theorem implies that $(B, A, H, Z) = (B', A', H', Z)$. Then (by the Fundamental Theorem), lines c and c' are projective lines with a self-corresponding point Z. That makes the projectivity a perspectivity, and hence the lines joining corresponding points are concurrent, i.e., lines BB', AA' and $HH' = CC'$ are concurrent. ☐

It remains to prove Pappus' Theorem and Theorems 3 and 4.

Pappus' Theorem The cross ratio of four lines of a pencil of lines equals the cross ratio of the four points at which an arbitrary line cuts the lines. (From *Collectiones mathematicae*, Pappus of Alexandria, fourth century A.D.)

Proof. Let A, B, C, D be the four points of intersection of a line with rays $OA = a$, $OB = b$, $OC = c$, $OD = d$. We designate the sine of the angle formed by two rays, for example, from a to c as $\sin ac$.



Since the perpendiculars from A and B to c have lengths $a \sin ac$ and $b \sin bc$, and are in the same ratio as AC to BC, we obtain

$$\frac{a \sin ac}{b \sin bc} = \frac{AC}{BC}.$$

Similarly,

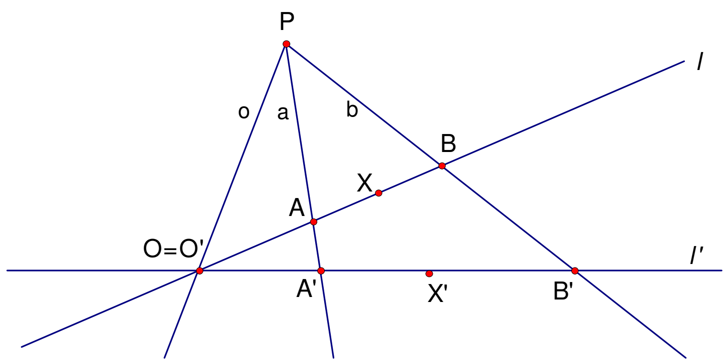
$$\frac{a \sin ad}{b \sin bd} = \frac{AD}{BD}.$$

(Directed distances and angles should be used here.) Division yields the desired result:

$$(a,b,c,d) = \frac{a \sin ac}{b \sin bc} \div \frac{a \sin ad}{b \sin bd} = \frac{AC}{BC} \div \frac{AD}{BD} = (A,B,C,D). \qquad \square$$

Theorem 3. If in a projectivity between two lines the point of intersection corresponds to itself, the projectivity is a perspectivity.

Proof. Let the two lines be l and l', and $O = O' = l \cap l$ be their point of intersection. Pick two fixed points A and B, and an arbitary point X on l. Let A', B', X' be their corresponding points on l'.



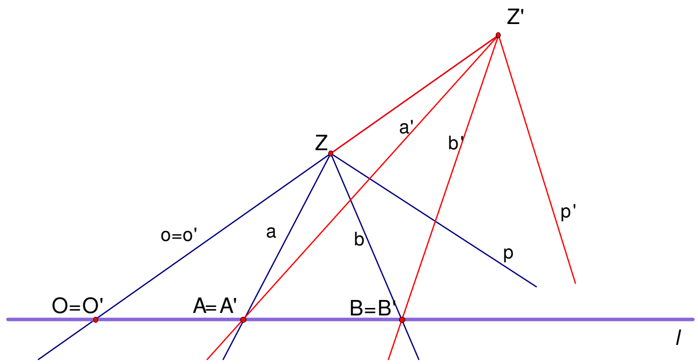
Let P be the point of intersection of lines AA' and BB', $o = PO$, $a = PA$, $b = PB$, $x = PX$ and $x' = PX'$. By Pappus' Theorem, $(o,a,b,x) = (O,A,B,X)$ and $(o,a,b,x') = (O',A',B',X')$, and since the right hand sides of these equations are equal (because the mapping between l and l' is a projectivity), it follows that

$$(o,a,b,x) = (o,a,b,x').$$

But if two equal cross ratios agree in three elements, then they must agree in the fourth. Consequently $x' = x$, and XX' passes through P which means the correspondence between l and l' is a perspectivity centered at P. $\square$

Theorem 4. If in a projectivity between two pencils of lines the line joining the two centers corresponds to itself, the projectivity is a perpsectivity.

Proof. Let the two centers of the projective pencils be Z and Z', and $o = o' = ZZ'$ be the self-corresponding line. Pick two lines a and b, and an arbitary line p through Z. Let a', b', p' be their corresponding lines through Z'.



Let l be the line through the points of intersection $a \cap a', b \cap b',\ O, A, B$ as shown in the figure, and $P = p \cap l$ and $P' = p' \cap l$. By Pappus' Theorem, $(o,a,b,p) = (O,A,B,P)$ and $(o,a',b',p') = (O',A',B',P') = (O,A,B,P')$, and since the left hand sides of these equations are equal (because the mapping between the pencils is a projectivity), it follows that

$$(O,A,B,P) = (O,A,B,P').$$

Consequently $P = P'$, and lines p and p' intersect on l. Thus the mapping is a perspectivity with axis l. □