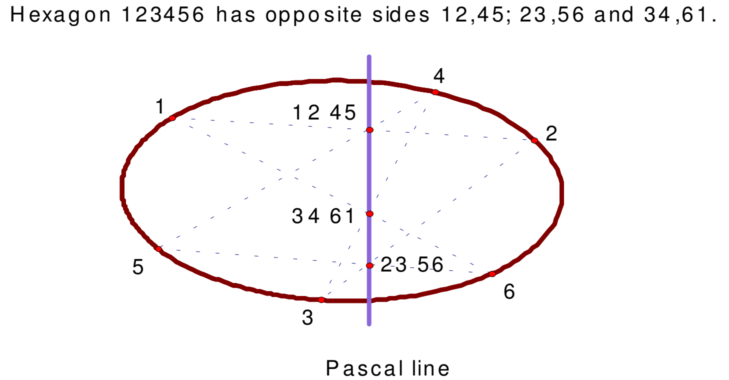
61. Pascal's Hexagon Theorem.

Prove that the three points of intersection of the opposite sides of a hexagon inscribed in a conic section lie on a straight line.



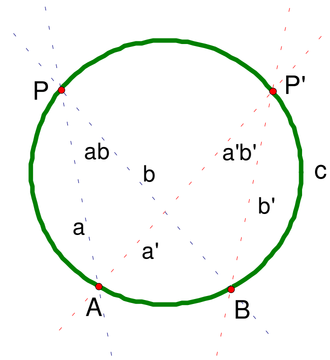
A hexagon inscribed in a conic section essentially consists of six points $1, 2, 3, 4, 5, 6$ anywhere on the conic section. The points are called "vertices" of the hexagon, and the segments (extended if necessary to get points of intersection) $12, 23, 34, 45, 56, 61$ are called the "sides" of the hexagon. The pairs 12 and 45, 23 and 56, and 34 and 61 are the "opposite sides" The straight line on which the three points of intersection lie is called the *Pascal line*. The hexagon itself is called a *Pascal hexagon*.

This fundamental theorem was published in 1640 by Blaise Pascal (1623-1662) at the age of 16 in his six-page paper *Essai sur les Coniques*. There are numerous proofs of Pascal's Theorem. The following projective proof is based on two theorems of Steiner:

Theorem 1. Any two points on a conic are centers of projective pencils of lines, of which the intersection points of corresponding lines produce all points on the conic.

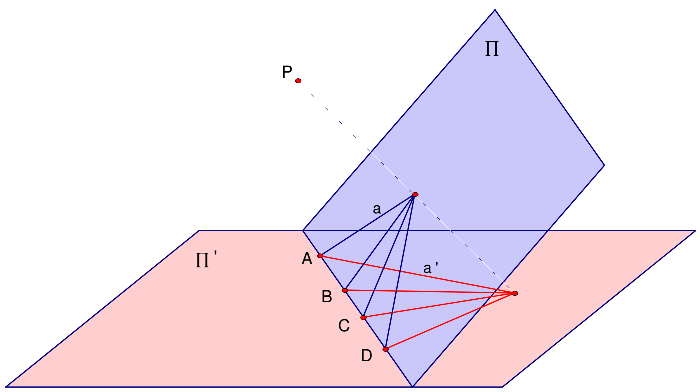
Theorem 2. If in a projectivity between two lines the point of intersection corresponds to itself, the projectivity is a perspectivity. (Theorem 3 from No. 59)

Proof of 1. First of all the result is true for a circle: let P and P' be two fixed (but arbitrary) points on circle c, and let lines a and a' correspond if and only if they intersect at a point A on c. This is a projective correspondence between projective the pencils of lines centered at P and P', the intersection of corresponding lines producing the circle:



$(a,b,c,d) = (a',b',c',d')$ because the angles ab and $a'b'$, ac and $a'c'$, etc. are equal inscribed angles in c, and thus the correspondence is projective.

Now since a conic section is a central projection of a circle (from the vertex of a cone), and since a central projection preserves pencils of lines, we need only show that a pencil of lines in one plane and its central projection in another plane are projective pencils. (Then the projective pencils to the circle at P and P' will project onto projective pencils to the conic section, corresponding pairs of which produce all the points on the conic.)

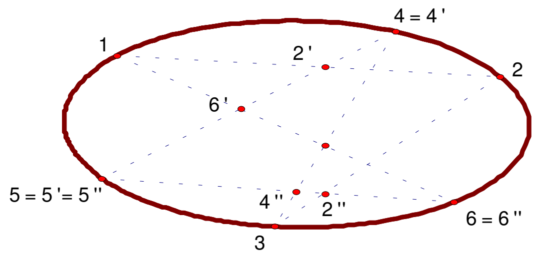


This follows from Pappus' Theorem (from No.59). Specifically, if a,b,c,d are four lines of a pencil in plane Π, their central projections in plane Π' are a',b',c',d' respectively, and their points of intersection on the line of intersection of the planes are $A = a \cap a', B = b \cap b'$, etc., then by Pappus' Theorem,

$$(a,b,c,d) = (A,B,C,D) = (a',b',c',d')$$

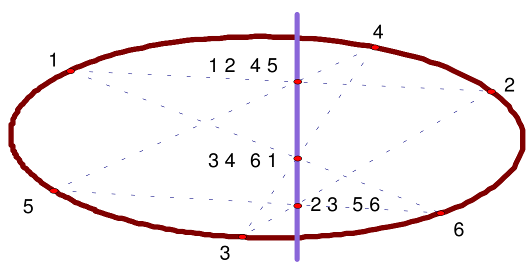
i.e., the pencil and its image are projective. □

Proof of Pascal's Theorem. Let the vertices of the hexagon be $1,2,3,4,5,6$. By Theorem 1, the lines from 1 and 3 to the points $2,4,5,6$ are corresponding lines from projective pencils (centered at 1 and 3); thus the points of intersection $2',4',5',6'$ and $2'',4'',5'',6''$ of these lines with lines 54 and 56 are corresponding points on projective lines 54 and 56.



Because the projectivity between 54 and 56 has a self-corresponding point ($5' = 5''$), the projectivity is a perspectivity (by Theorem 2). Thus corresponding lines intersect at a point, i.e., $2'2''$, $4'4''$ and $6'6''$ intersect at a point, but $4'4'' = 34$ and $6'6'' = 61$. Call this point Z. Then $2', Z$ **and** $2''$ **are collinear.** This proves the theorem since

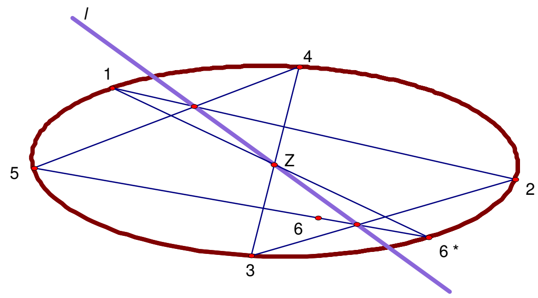
$$\begin{aligned} 2' &= 12\,45, \text{ the intersection of opposite sides } 12 \text{ and } 45, \\ Z &= 34\,61, \text{ the intersection of opposite sides } 34 \text{ and } 61, \\ 2'' &= 23\,56, \text{ the intersection of opposite sides } 23 \text{ and } 56. \end{aligned}$$



In the remainder of this section, we prove the converse of Pascal's Theorem and derive several corollaries from Pascal's Theorem.

The converse. If the opposite sides of a hexagon 123456 (of which no three vertices are collinear) intersect on a straight line, then the six vertices lie on a conic section.

Proof. The proof is by contradiction. There is a unique conic through points $1, 2, 3, 4, 5$. Suppose 6 is not on this conic.



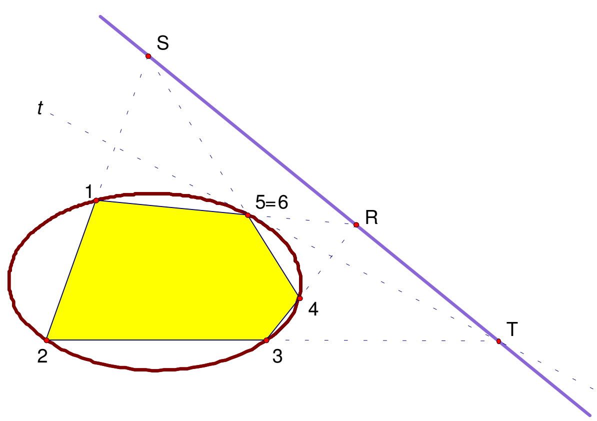
Let 6^* be the intersection point of the conic and line 56. By Pascal's Theorem, 6^* can be found in the following way:

1. Find the Pascal line l, i.e., the line through points $12\,45$ and $23\,56$. (Note that 6^* is on line 56, so $56 = 56^*$, and we can find point $23\,56$ even if we don't know 6^*.)
2. Let $Z = l \cap 34 = 34 \cap 6^*1$ (so $Z, 6^*, 1$ are collinear).
3. $6^* = 1Z \cap 56.$

On the other hand $Z = l \cap 34 = 34 \cap 61$ (by hypothesis) and so $6 = 1Z \cap 56$. This contraction proves the converse. □

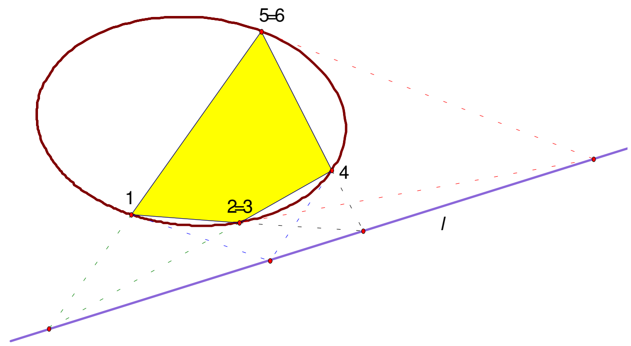
The following corollaries follow from Pascal's Theorem by considering what happens when vertices of a Pascal hexagon coincide.

Corollary 1. In every pentagon inscribed in a conic section, the points of intersection of two pairs of nonadjacent sides and the point of intersection of the fifth side with the tangent passing through its opposite vertex lie on a straight line.



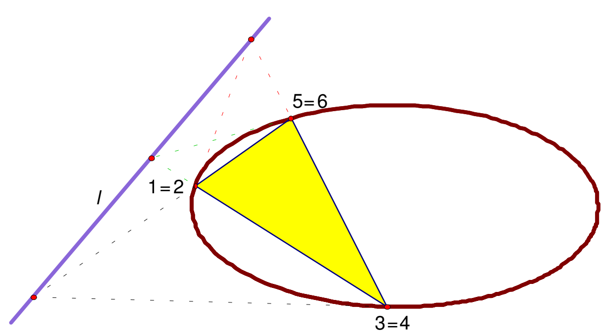
Proof. As $6 \to 5$, chord $56 \to$ the tangent line at 5, which we denote by t. By Pascal's Theorem, the points of intersection of opposite sides $S = 12 \cap 45$, $T = 23 \cap t$, and $R = 34 \cap 15$ are collinear. □

Corollary 2. In every quadrilateral inscribed in a conic section, pairs of opposite (extended) sides and tangents to the pairs of opposite vertices intersect on a straight line.



Proof. Denote opposite vertices of the quadrilateral as coincident vertices of a hexagon, say $2 = 3$ and $5 = 6$. Opposite sides are $12, 45$ and $24, 51$. Opposite sides 23 and 56 become tangent lines to the conic. They intersect on a line l. We could just as easily have chosen 1 and 4 to be the coincident vertices, and that puts the intersection of the tangent lines at 1 and 4 on l too. □

Corollary 3. In every triangle inscribed in a conic section, the sides intersect with the tangents to the opposite vertices on a straight line.



Proof. Denote the vertices of the triangle as coincident vertices of a hexagon, say $1 = 2$, $3 = 4$ and $5 = 6$. In Pascal's Theorem, "pairs of opposite sides" becomes "sides and tangents to opposite vertices", e.g. 12 (tangent line at 1) and 45. □