63. Desargues' Involution Theorem.

Note 1. For readers unfamiliar with projective geometry or unfamiliar with the somewhat dated terminology in Dörrie, this one is really hard to read. I intend to explain the result, but not all the details. The following is just a paraphrase of Dörrie. One needs to understand a few definitions to start with.

Definition

1. A **complete quadrangle** consists of 4 coplanar points $1, 2, 3, 4$, no three collinear, and the six lines $12, 13, 14, 23, 24, 34;$ the pairs 12 and $34,$ 13 and $24,$ and 14 and 23 are called **opposite sides**.

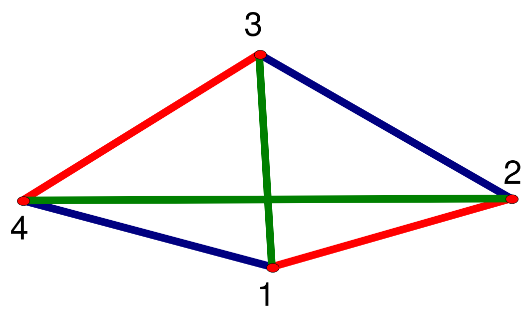

Figure 1.

2. An **involution** is a non-identity projectivity between the points on a line (or a circle) or the lines through a point whose square is the identity, i.e., $\sigma : \ell \to \ell$ is an involution if $\sigma(A) \neq A$ for some point A on ℓ, but $\sigma(\sigma(A)) = A$ for all A on ℓ; if $\sigma(A) = A'$, then $\sigma(A') = A$. A and A' (or A' and A) correspond reciprocally, are reciprocal points, or are a **reciprocal pair**.

Desargues' Theorem. The (eight) points of intersection of a line ℓ with the three pairs of opposite sides of a complete quadrangle and a conic section circumscribed about the quadrangle form four pairs of an involution on ℓ.

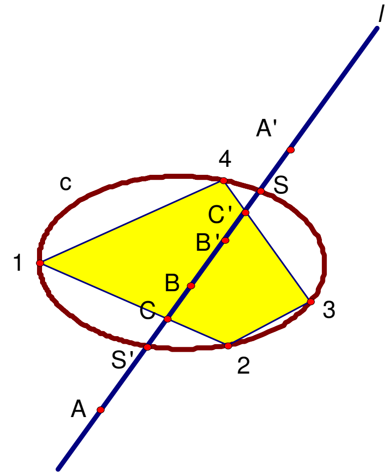

Figure 2.

In the figure above $(A,A'),(B,B'),(C,C')$ and (S,S') are reciprocal pairs of an involution on ℓ.

The proof will be given later on.

Special cases arise when some points of the quadrangle coincide.

Corollary 1. (Let $4 \to 3.$) The points of intersection of a straight line ℓ with

1. a conic section,
2. with two sides of a triangle inscribed in the conic section, and
3. with the third side of the triangle (extended if necessary) and the tangent line to the conic at the opposite vertex

are (three) reciprocal pairs of an involution on ℓ.

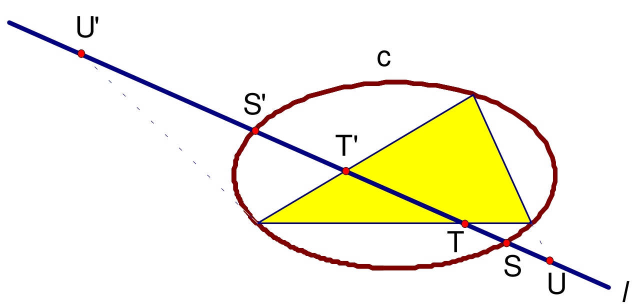

Figure 3.

i.e. $(S,S').(T,T')$ and (U,U') are reciprocal pairs.

Corollary 2. (Let $4 \to 3$ and $2 \to 1$.) The points of intersection of a straight line ℓ with

1. a conic section,
2. two tangents to the conic, and
3. the chord joining the points of tangency

are reciprocal pairs of an involution on ℓ in which the (single) point in 3 is a fixed point of the involution.

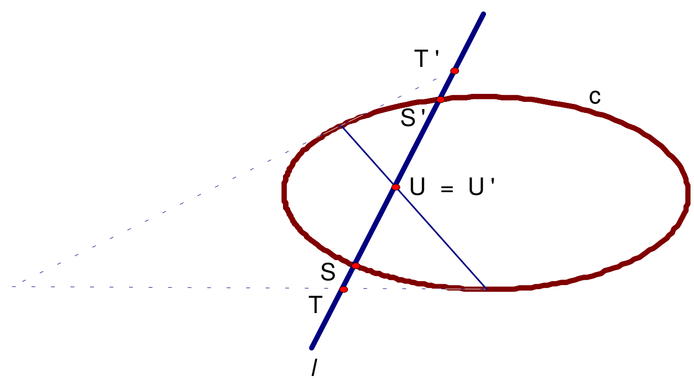

Figure 4.

i.e. (S,S') and (T,T') are reciprocal pairs of an involution on ℓ with fixed point U.

Several preliminary results are needed to prove Desargues' Theorem:

Theorem 1. A projectivity σ between points on a line ℓ which has a reciprocal pair (A,A') (with $A \neq A'$) is an involution.

Proof. Let $\sigma(P) = P' = Q$ and $\sigma(Q) = Q'$. Then $(A,A',P,Q) = (A',A,P',Q') = (A',A,Q,Q')$, but $(A,A',P,Q) = (A',A,Q,P)$ (by a property of cross ratios), and it follows that $(A',A,Q,Q') = (A',A,Q,P)$ so $Q' = P$, i.e. $\sigma(P') = P$. Thus the projectivity σ is an involution. $\square$

Theorem 2. Any two points on a conic are centers of projective pencils of lines, of which the intersection points of corresponding lines produce all points on the conic. (This is Theorem 1 from No. 61.)

Proof of Desargues' Theorem. (See Figure 2 above.) Let quadrangle 1234 be inscribed in conic c. Let line ℓ intersect opposite sides 23 and 14 in A and A' respectively, 31 and 24 in B and B', 12 and 34 in C and C' and c at S and S'. The pencils of lines through centers 1 and 4 are projective with lines $12, 13, 1S, 1S'$ corresponding to $42, 43, 4S, 4S'$ respectively (by Theorem 2). This gives a projectivity σ on l with $CBSS' \barwedge B'C'SS'$, and thus $(C,B,S,S') = (B',C',S,S')$. By properties of the cross ratio, $(B',C',S,S') = (C',B',S',S)$ so $(C,B,S,S') = (C',B',S',S)$ and $CBSS' \barwedge C'B'S'S$ and by Theorem 1, σ is an involution (since (S,S') is a reciprocal pair) with $B \to B' \to B$ and $C \to C' \to C$. We could just as easily used points 2 and 4 on the conic with points

$3, 1, S', S$ to get $(A, C, S, S') = (A', C', S', S)$ and conclude that (A, A') is also a reciprocal pair. □

Note 2. There are dual statements for all the results above, obtained by interchanging "point" and "line" and making suitable changes in the language. Dörrie covers all these cases. In contemporary language, the dual of a complete quadrangle is a complete quadrilateral consisting of four distinct lines, no three of which are concurrent and their six points of intersection. "A conic circumscribed about a quadrangle" becomes "a conic inscribed in a quadrilateral".

Note 3. Dörrie also talks about involutions on a line and on a circle in some detail. These are interesting and useful results. He puts them before the proof the theorem and this makes things look more difficult than they really are. His discussion about involutions on lines uses Desargues' Theorem.

Involutions on a line. Let (A, A') and (B, B') be reciprocal pairs of points on line ℓ, and let C be any other point on ℓ. We construct a complete quadrangle 1234 as follows:

1. Draw lines through A, B, C forming triangle $\triangle 123$ with A on 23, B on 31 and C on 12,
2. Point 4 is the intersection of lines $1A'$ and $2B'$.

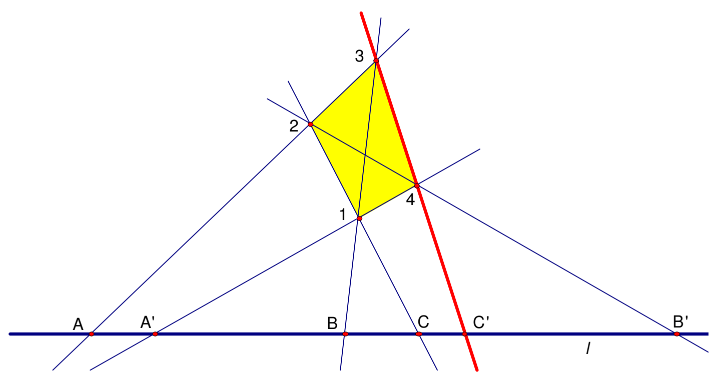

Figure 5.

By Desargues' Theorem, ℓ meets the opposite sides of 1234 in reciprocal pairs, two of which are (A, A') from 23 and 14, and (B, B') from 13 and 24. The third pair is (C, C') with $C' = \ell \cap 34$. (Note: conic sections are not needed.) □

The construction of the involution between two pencils of lines is done similarly.

Involutions on a circle. Let (A, A') and (B, B') be reciprocal pairs of points on circle c, and let C be any other point on c. The pencils centered at A and A' in which $AA', AB, AB' \leftrightarrow A'A, A'B', A'B$ is a projectivity. Since AA' corresponds to itself, this projectivity is a perspectivity. (See No. 59.) The axis of perspectivity is line ZO where $Z = AB \cap A'B'$ and $O = AB' \cap A'B$.

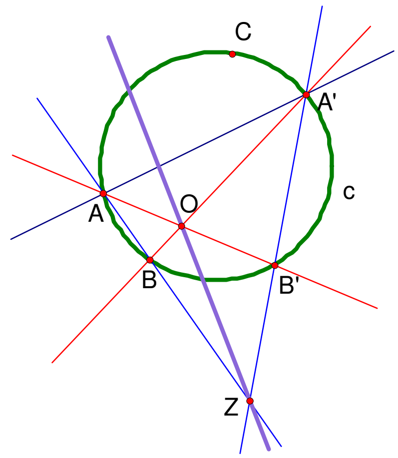

Figure 6.
C' is that point on c at which AC and $A'C'$ intersect on the axis. Thus if $Y = ZO \cap AC$, $C' = c \cap A'Y$:

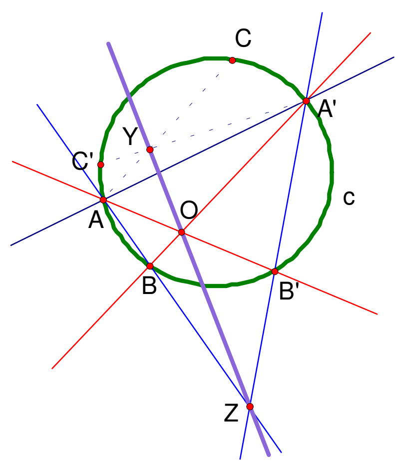

Figure 7.

The argument can be carried out with pencils centered at B and B' (instead of A and A'), and then $C' = c \cap B'X$ where $X = ZO \cap BC$.

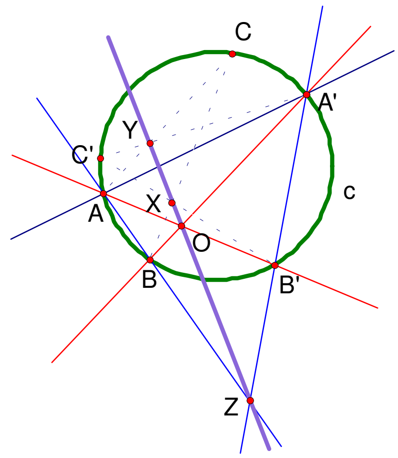

Figure 8.

$\triangle ABC$ and $\triangle A'B'C'$ are perspective from line ZO, and by Desargues' two triangle theorem (No. 59), they are perspective from a point, i.e., lines AA', BB' and CC' are concurrent, say at point S. Any secant line of c through S cuts c in a reciprocal pair of points of the involution. (This gives a quick way to find C'.)

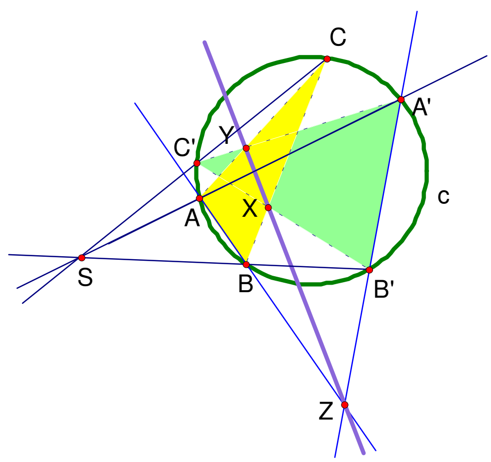

Figure 9.

We conclude that

Theorem 3.

1. The lines joining reciprocal points of an involution on a circle pass through a fixed point.
2. A secant rotated about a fixed point cuts a circle in a pair of reciprocal points of an involution.

and since every conic section is the central projection of a circle (from the vertex of a cone), and since projectivities (including involutions) are preserved by projection (by Pappus'

Theorem, No. 59),

Theorem 4.

1. The lines joining reciprocal points of an involution on a conic section pass through a fixed point.
2. A secant rotated about a fixed point cuts a conic section in a pair of reciprocal points of an involution.

Note 4. The fixed point in Theorems 3 and 4 may be outside or inside the circle or conic section.

Note 5. There are dual versions of both theorems. For example, the dual of Theorem 4 is

Theorem 4'.

1. The points of intersection of reciprocal tangents of an involution on a conic section lie on a fixed line.
2. The tangents from points on a fixed line to a conic section are reciprocal pairs of an involution.

Note 5. Dörrie concludes with the comment that there are infinitely many conic sections that pass through the four vertices of a complete quadrangle, (including the opposite sides as degenerate conic sections). These conic sections constitute a pencil of conic sections. Then Desargues' Theorem can be expressed succinctly in the following way.

Desargues' Involution Theorem. A line ℓ intersects the conic sections of a pencil of conic sections in reciprocal pairs of an involution on ℓ. (There is a dual statement as well.)