64. A Conic Section from Five Elements.

To draw a conic section C of which five elements - points and tangents - are known.

We consider the three cases:

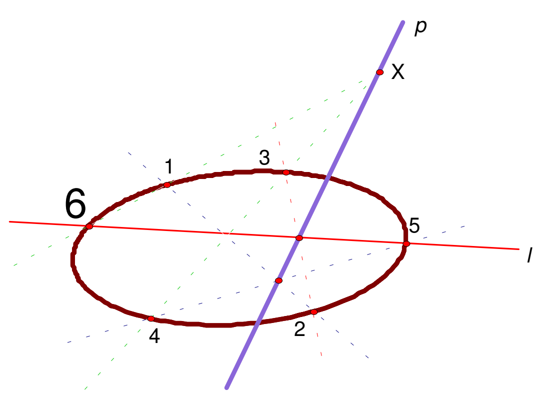
1. Five points are known.
2. Four points and a tangent line are known.
3. Three points and two tangents are known.

Note 1. Dörrie considers other cases, which are essentially "duals" of these cases. For example the case of two known points and three known tangent lines is the dual of 3 above. The proofs of 1,2 and 3 rely on Pascal's Theorem (No. 61), and the proof of the duals on the dual of Pascal's Theorem, namely Brianchon's Theorem (No. 62).

Note 2. Most geometry software programs include a tool to construct a conic through 5 points. This is case 1 above.

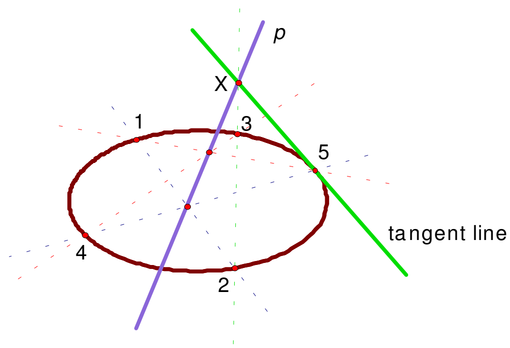
1. Let the five points be $1,2,3,4,5$ in any order.

Let ℓ be a line through point 5. We will determine point 6 on ℓ and on C. First find the Pascal line p as the line through the points of intersection of opposite sides 12 and 45, and 23 and $56 = \ell$. Lines 34 and 61 also intersect on p, say at X; then $6 = \ell \cap 1X$.



Different lines ℓ through 5 will produce different points 6.

It is also possible to construct the tangent line to C at any of the five given points, say 5. This uses Corollary 1 to Pascal's Theorem. Let p be the line through points $12 \cap 45$ and $34 \cap 51$. Let $X = p \cap 23$. Then line $5X$ is tangent to C at 5.

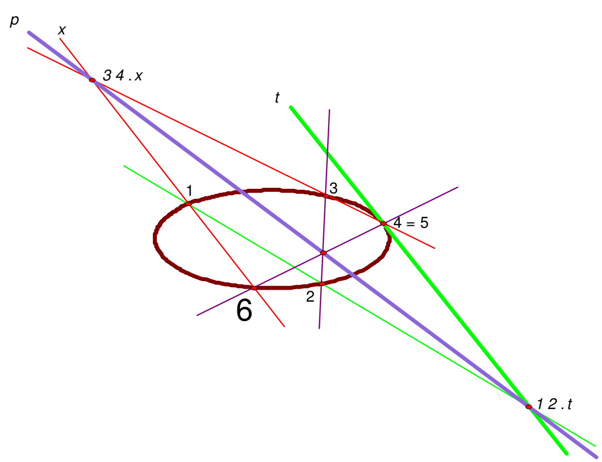


2. Draw a conic section of which four points $1, 2, 3, 4$ and one tangent t are given. There are two subcases to consider.

a. The tangent t passes through one of the given points, say 4.

We consider the tangent t as the line connecting two coincident points 4 and 5, so that $t = 45$. Let 6 be the point of intersection of conic C with an arbitrary line x through 1. Thus $x = 16$.

Now draw the Pascal line p of hexagon 123456 as the line through the points of intersection of 12 and $45 = t$, and 34 and $61 = x$.



The point of intersection of 23 and $56 = 46$ lies on p, and it follows that 6 is the point of intersection of x and the line through 4 and $23 \cap p$.

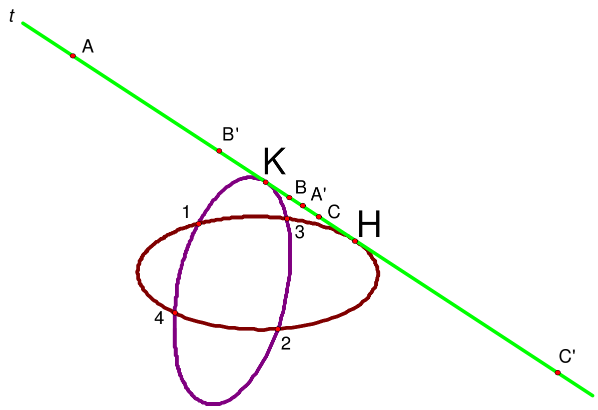
$1, 2, 3, 4 = 5$ and 6 are then five known points on conic C, and the problem is now reduced to case 1.

b. The tangent t does not pass through any of the given points.

To solve this problem, we use Desargues' involution theorem (No. 63) with the complete quadrangle 1234 and line t. Let A, A', B, B' be the points of intersection of t with opposite sides $12, 34, 23, 41$ respectively. The reciprocal pairs (A, A') and (B, B') determine an involution on t (No.

63) A third reciprocal pair (C, C') can be found by intersecting opposite sides 13 and 24 with t, and then the double points H and K of the involution can be constructed by Steiner's construction (No. 60). These are points of tangency of t on the conic.

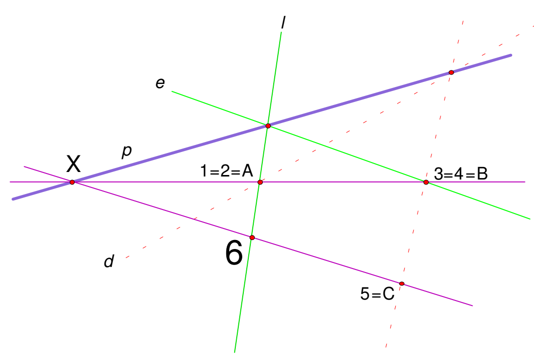
Five points are then known on conic C, and the problem is reduced to case 1. There are generally two conics. If the involution has no double points, there are no solutions.



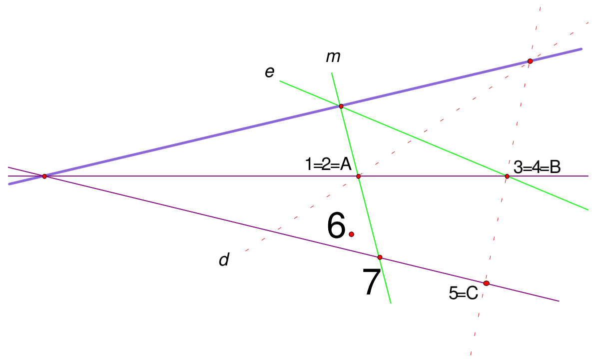
3. Draw a conic section of which three points A, B, C and two tangents d and e are given. There are three subcases to consider.

a. d passes through A and e passes through B.

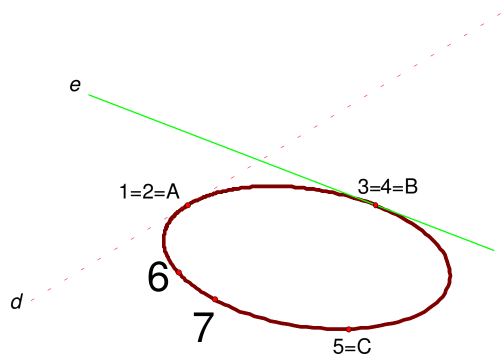
Construct the Pascal line p of the "hexagon" 123456 in which $1 = 2 = A$, $3 = 4 = B$, $5 = C$ and 6 is on an arbitrary line ℓ through A. Note that sides 12 and 34 are the tangent lines d and e respectively. p is the line through the points of intersection of $12 = d$ and $45 = BC$, and $34 = e$ and $61 = \ell$. The point of intersection X of $23 = AB$ and $56 = C6$ must be on ℓ, and it follows that $6 = 5X \cap \ell$.



Similarly, with a different line m through A, construct point 7 on the conic.

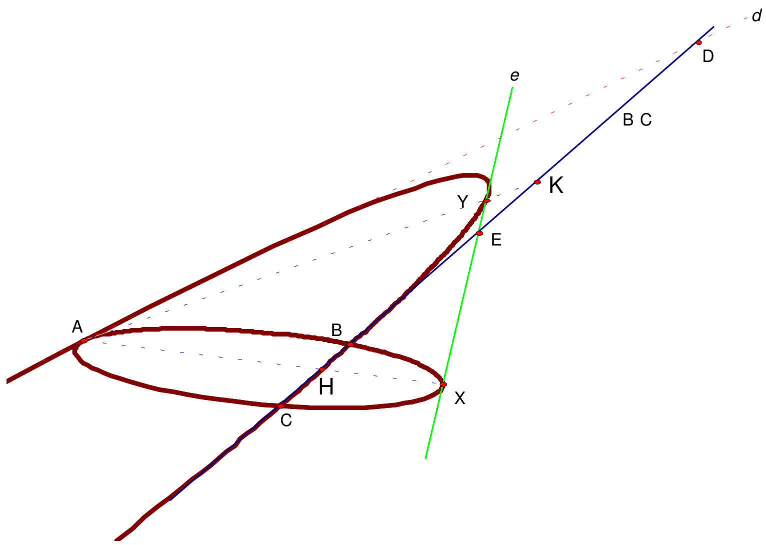


Now that five points on the conic are know, the conic can be "constructed" as in case 1.



b. d passes through A, but e does not pass through any of the given points.

The solution uses the second corollary to Desargues' involution theorem. (See No. 63.) Let $D = BC \cap d$ and $E = BC \cap e$. Construct the double points H and K of the involution determined by the reciprocal pairs (B, C) and (D, E). Lines AH and AK intersect line e at points of tangency X and Y on the conic, and the problem reduces to subcase a above.



There are no solutions if the involution has no double points.

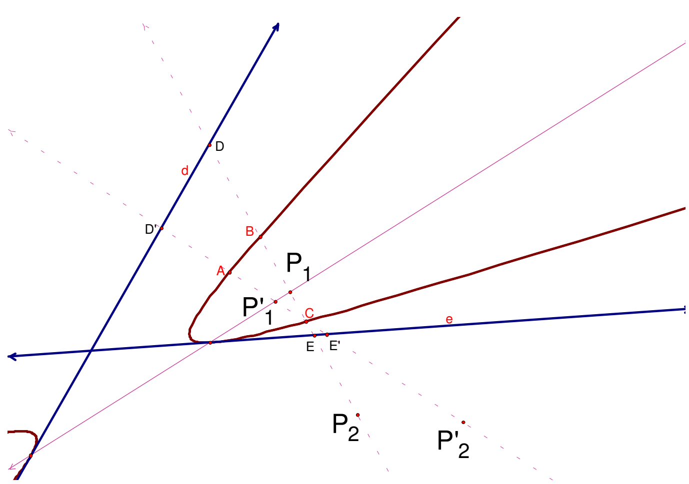
c. Neither of the two tangents passes through any of the given points.

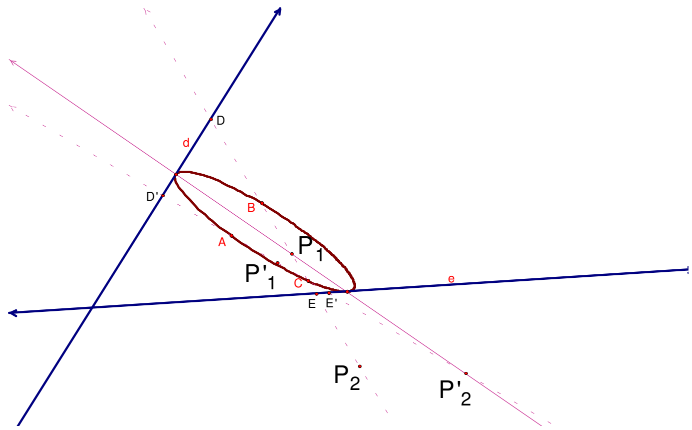
Let $D = d \cap BC$ and $E = e \cap BC$. Determine a double point P of the involution determined by the reciprocal pairs (B, C) and (D, E). P lies on the chord joining points of tangency (on C) of d and e (by the second corollary to Desargues' Involution Theorem).

Let $D' = d \cap CA$ and $E' = e \cap CA$. Determine a double point P' of the involution determined by the reciprocal pairs (C, A) and (D', E'). P' lies on the chord joining points of tangency of d and e.

The line PP' is thus the tangency chord of the last two paragraphs, and it meets d and e at their tangency points. This gives five points on conic C, and the conic can be "constructed" as in case 1.

Both involutions must have a double point in order for C to exist. In this case, there are in fact four solutions, since each involution has two double points

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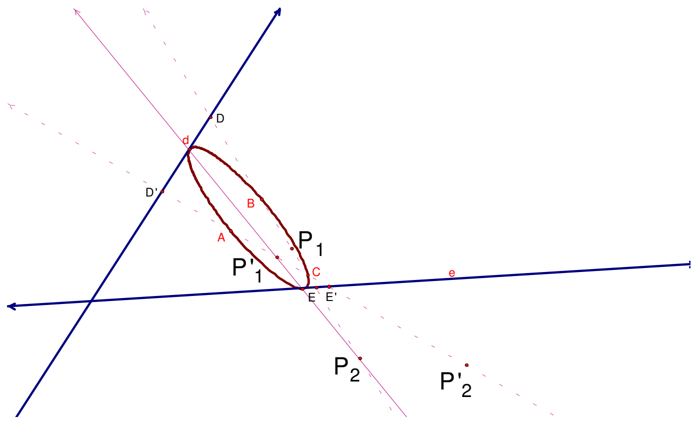
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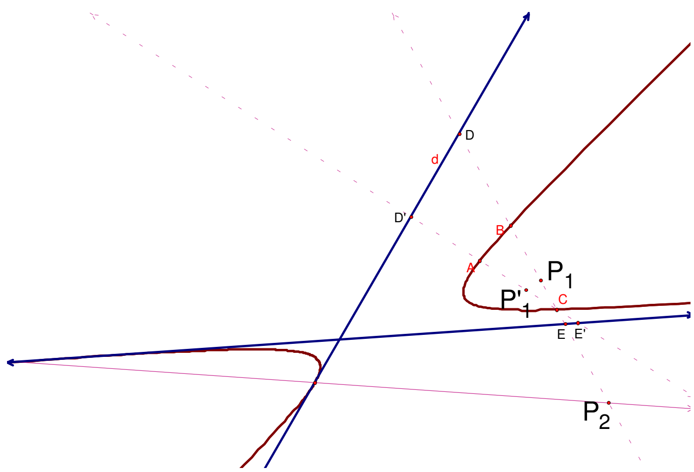
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