10. Cauchy's Mean Theorem

The geometric mean of several positive number is less than or equal to the arithmetic mean of these numbers, i.e., $\sqrt[n]{abc\ldots} \le \frac{a+b+c+\ldots}{n}$ **with equality holding only when all** n **numbers** $a, b, c, \ldots$ **are equal.**

Augustin Louis Cauchy (1789-1857) was one of the greatest French mathematicians. This theorem about the arithmetic and geometric means occurs in his *Cours d'Analyse* (pp. 458-9), which was published in 1821.

The proof of the theorem presented here is based on the solution of the following problem: **When does the product of** n **positive numbers of constant sum attain its maximum value**?

Let the n numbers be $a, b, c, \ldots,$ their constant sum be $K,$ and their product $P.$ Experimentation suggests that P reaches it maximum value when the $a = b = c = \ldots = M = \frac{K}{n}.$ We begin by proving the

Lemma Of two pairs of numbers with equal sum, the pair with the smaller difference has the larger product.

Proof Let the pairs be X, Y and $x, y,$ so that $X + Y = x + y = K$ is constant. Note that

$$4XY = (X+Y)^2 - (X-Y)^2, \qquad 4xy = (x+y)^2 - (x-y)^2$$

from it follows that the larger product occurs for the pair with the smaller difference. The product is largest when $X = Y = \frac{K}{2}.$ ☐

We move on to $n > 2$ numbers now. If the n numbers $a, b, c, \ldots$ are not all equal, then at least one, say $a,$ must be greater than the average $M,$ and at least one, say b is less than $M.$ Form a new system of n numbers $a', b', c', \ldots$ where $a' = M,\ a + b = a' + b'$ and $c' = c,\ d' = d,$ etc. The new numbers have the same sum K as the old ones, but a greater product by the Lemma, since $a' - b' < a - b.$

If the numbers $a' = M, b', c', \ldots$ are not all equal to $M,$ then at least one of them, say b' is greater (smaller) than $M,$ and at least one of them, say c' is smaller (greater) than $M.$ Form a new system of n numbers $a'', b'', c'', d'', \ldots$ in which $a'' = a' = M,\ b'' = M,$ $b' + c' = b'' + c'',$ and $d'' = d',\ e'' = e',$ etc. These new number have the same sum K as $a', b', c', \ldots$ but a greater product by the Lemma, since $|b'' - c''| < |b' - c'|.$

Continue in this way to get a sequence of increasing products, each member of which is greater than the one before it by at least one more multiple of the factor $M.$ The last product obtained in this manner is the greatest of all, and consists of n equal factors of $M.$ Thus $P \le M^n$ which proves the

Theorem The product of n **positive numbers with constant sum is largest when all the numbers are equal.**

Extracting n^{th} roots in $P \leq M^n$, we obtain *Cauchy's formula*:

$$\sqrt[n]{abc\ldots} \leq \frac{a+b+c+\ldots}{n}.$$

or

Theorem of the Arithmetic and Geometric Mean: **The geometric mean of several positive number is less than or equal to the arithmetic mean of these numbers, with equality holding only when all the numbers are equal.**

Note 1. Cauchy's Theorem leads directly to the following

Theorem The sum of n positive numbers with a constant product is minimal when the numbers are equal.

Proof. Let the n numbers be $x, y, z, \ldots,$ $k = xyz\ldots,$ and s their variable sum. Also let $m = \sqrt[n]{k}$. By Cauchy's Theorem, $\frac{x+y+z+\ldots}{n} \geq \sqrt[n]{xyz\ldots}$, i.e., $\frac{s}{n} \geq m$ or $s \geq mn$. Thus s is always at least mn, and equal to mn only when $x = y = z = \ldots = m$. ☐

The preceding two extreme theorems form the basis for simple solutions to many max/min problems, e.g., Nos. 54, 92, 96, 98.

Note 2. Cauchy's theorem also provides us with a proof of the exponential inequality for positive rational exponents less than 1, i.e.,

Theorem Let ε be a positive rational number less than 1, and a any positive real number. Then $a^{\varepsilon} \leq 1 + \varepsilon(a-1)$.

Proof. Let $\varepsilon = \frac{n}{m}$ with $0 < n < m$. Consider m numbers, n of which are equal to a, and $m - n$ of which are equal to 1. Their geometric mean is $\sqrt[m]{a^n} = a^{\frac{n}{m}} = a^{\varepsilon}$, and their arithmetic mean is $\frac{na+m-n}{m} = 1 + \frac{n}{m}(a-1) = 1 + \varepsilon(a-1)$, and the desired inequality follows from Cauchy's theorem. ☐

Note 3. Dörrie argues that $a^{\varepsilon} \leq 1 + \varepsilon(a-1)$ is also true for any positive real number $\varepsilon < 1$, and considers the case $\varepsilon > 1$ too. Today, it is easier to do this using Calculus. We have

The Exponential Inequality Let $x > 0$. Then

$$x^c \leq 1 + c(x-1) \quad \text{if } 0 < c < 1$$
$$x^c \geq 1 + c(x-1) \quad \text{if } c > 1$$

with equality holding if and only if $x = 1$.

Proof. Let $0 < c < 1$, and $f(x) = x^c - [1 + c(x-1)]$ for $x > 0$.
$f'(x) = cx^{c-1} - c = c\left(\frac{1}{x^{1-c}} - 1\right)$. The only critical number is $x = 1$. If $x < 1$, then $f'(x) > 0$ and if $x > 1$, $f'(x) < 0$. This establishes that f has a minimum at $x = 1$, i.e., $f(x) \geq f(1) = 0$ with equality if and only if $x = 1$. The proof in the case of $c > 1$ is similar in that it examines the sign of $c(x^{c-1} - 1)$. [Of course, one must grant the

validity of the power rule $\frac{d}{dx}x^c = cx^{c-1}$ for all $c > 0$.] □