11. Bernoulli’s Power Sum Problem

Determine the sum

\[ S = 1^p + 2^p + 3^p + \cdots + n^p \]

of the \( p^{th} \) powers of the first \( n \) natural numbers for positive integer exponents \( p \).

The problem, posed in this general form, was first solved in "Ars Conjectandi" ("The Art of Conjecture" an early work on probability) in 1713 by the Swiss mathematician Jacob Bernoulli (1654-1705).

The following elegant solution is based on the binomial theorem. Let \( P = p + 1 \). Then

\[
(v + S)^p = v^p + P v^p S + \binom{p}{2} v^{p-1} S^2 + \ldots \quad \text{and} \quad (v + S - 1)^p = v^p + P v^p (S - 1) + \binom{p}{2} v^{p-1} (S - 1)^2 + \ldots
\]

and subtraction gives us

\[
(l) \quad (v + S)^p - (v + S - 1)^p = P v^p + \binom{p}{2} v^{p-1} \left[ S^2 - (S - 1)^2 \right] + \binom{p}{3} v^{p-2} \left[ S^3 - (S - 1)^3 \right] + \ldots
\]

Making the substitution \( S_v \) for \( S^v \) in the equations

\[
(1) \quad (S - 1)^2 = S^2 \\
(2) \quad (S - 1)^3 = S^3 \\
(3) \quad (S - 1)^4 = S^4
\]

etc.

gives us \(-2S_1 + 1 = 0\) and \( S_1 = \frac{1}{2} \), \(-3S_2 + 3S_1 - 1 = 0\) and \( S_2 = \frac{1}{6} \), \(-4S_3 + 6S_2 - 4S_1 + 1 = 0\) and \( S_3 = 0 \). Similarly \( S_4 = \frac{-1}{30} \). These are the Bernoulli numbers.

Then from (l), we conclude that

\[
P v^p = \left[ (v + S)^p - (v + S - 1)^p \right]_{S=S^v}
\]

or just

\[
(la) \quad P v^p = (v + S)^p - (v + S - 1)^p,
\]

the substitutions being understood. Now let \( v = 1, 2, 3, \ldots, n \) to get \( n \) equations:
\[ P \cdot 1^p = (1 + S)^p - S^p \]
\[ P \cdot 2^p = (2 + S)^p - (1 + S)^p \]
\[ \vdots \]
\[ P \cdot n^p = (n + S)^p - (n - 1 + S)^p, \]

addition of which gives us

\[ (II) \quad PS = (n + S)^p - S^p \]

or

\[ (II) \quad 1^p + 2^p + 3^p + \cdots + n^p = \frac{(n+S)^p - S^p}{p} l_{S, S^p}. \]

Using the \( S \) \(_n\) values found above, we obtain the following results:

\[ 1 + 2 + 3 + \ldots + n = \frac{(n+S^2 - S^2)}{2} l_{S, S^p} \]
\[ = \frac{n^2 + 2nS_1}{2} \]
\[ = \frac{n^2 + n}{2} \]
\[ = \frac{n(n+1)}{2}, \]

\[ 1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{(n+S^3 - S^3)}{3} l_{S, S^p} \]
\[ = \frac{n^3 + 3n^2S_1 + 3nS_2}{3} \]
\[ = \frac{2n^3 + 3n^2 + n}{6} \]
\[ = \frac{n(n+1)(2n+1)}{6}, \]

\[ 1^3 + 2^3 + 3^3 + \ldots + n^3 = \frac{(n+S^4 - S^4)}{4} l_{S, S^p} \]
\[ = \frac{n^4 + 4n^3S_1 + 6n^2S_2 + 4nS_3}{4} \]
\[ = \frac{n^4 + 2n^3 + n^2}{4} \]
\[ = \left[ \frac{n(n+1)}{2} \right]^2, \]
\[ 1^4 + 2^4 + 3^4 + \ldots + n^4 = \frac{(n+1)^5 - S^5}{5}|_{S=n}^{S=n} = \frac{n^5 + 5n^4 S + 10n^3 S^2 + 10n^2 S^3 + 5n S^4}{5} = \frac{6n^5 + 15n^4 + 10n^3 - n}{30} = \frac{n(n+1)(2n+1)\left(3n^2 + 3n - 1\right)}{30}. \]

\[ PS = (n + S)^p - S^p = n^p + \binom{p}{1} S_1 n^{p-1} + \binom{p}{2} S_2 n^{p-2} + \ldots \]
can be rewritten as

\[ \frac{S}{n^p} = \frac{1}{p} + \frac{\binom{p}{1} S_1}{Pn} + \frac{\binom{p}{2} S_2}{Pn^2} + \ldots \]

which as \( n \to \infty \) appears to approach \( \frac{1}{p} = \frac{1}{p+1} \). [One should know something about the long-term behavior of \( S \) to be sure of this.] This suggests that the average value of \( 1^p, 2^p, \ldots, n^p \) is \( \frac{1}{p+1} \). One application of the definite integral in an elementary calculus course is to find the average value of a function. More specifically, the average value of \( f(t) \) on the interval \([a, b]\) is \( \frac{1}{b-a} \int_a^b f(t)dt \). It follows that the average value of \( t^p \) from 0 to \( x \) is

\[ \frac{1}{p+1} \int_0^x t^p dt = \frac{1}{p+1} \left( \frac{x^{p+1}}{p+1} \right) = \frac{x^p}{p+1}, \]

agreeing with the result above, and extending it to any \( p > 0 \).