13. Newton's Exponential Series

Find the power series representation for e^x.

The exponential series, which may be the most important series in mathematics, was discovered by the great English mathematician and physicist Isaac Newton (1642-1727). The famous treatise that contains the sine series, the cosine series, the arc sine series, the logarithmic series and the binomial series as well as the exponential series was written in 1665 and bears the title *De analysi per aequationes numero terminorum infinitas*. Newton's derivation of the exponential series, is however, not rigorous and rather complicated.

The following derivation is based on the mean values of the functions x^c and e^x. [Today, it's common to use Maclaurin's formula $\sum_{n=0}^{\infty} f^{(n)}(0)\frac{x^n}{n!}$ to find the power series expansion for $f(x) = e^x$. $f^{(n)}(x) = e^x$, and $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ for all real numbers x.] The mean (average) values of x^c and e^x from 0 to x are

$$\frac{1}{x}\int_0^x t^c\,dt = \frac{x^c}{c+1}$$
$$\frac{1}{x}\int_0^x e^t\,dt = \frac{e^x-1}{x}$$

for both positive and negative values of x. (For negative x, the average value of e^x from x to 0 is $\frac{1}{0-x}\int_x^0 e^t\,dt = \frac{1-e^x}{-x} = \frac{e^x-1}{x}$, and similarly for x^c.)

Start with the exponential inequality $e^x > 1+x$ (See No. 12), and find the average value of each side to conclude that $\frac{e^x-1}{x} > 1+\frac{x}{2}$ and for $x>0$, $e^x > 1+x+\frac{x^2}{2!}$. Repetition of this argument (for $x>0$) leads to $e^x > 1+x+\frac{x^2}{2!}+\frac{x^3}{3!}$ and more generally

(1) $$e^x > 1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots+\frac{x^n}{n!}.$$

To obtain an upper limit for e^x, start with $e^{-x} > 1-x$, multiply by e^x to get $1 > e^x - xe^x$ or $e^x < 1+xe^x$. Note that if $v(x) \le M$ on an interval $[a,b]$, then the average value of $u(x)v(x)$ on $[a,b]$ is $\frac{1}{b-a}\int_a^b u(x)v(x)\,dx \le \frac{M}{b-a}\int_a^b u(x)\,dx = M$ •the average value of $u(x)$. Thus with $u(x) = x$ and $v(x) = e^x$, $\frac{e^x-1}{x} < 1+\frac{x}{2}e^x$ or $e^x < 1+x+\frac{x^2}{2!}e^x$ (for $x>0$) and then with $u(x) = \frac{x^2}{2}$ and $v(x) = e^x$, $\frac{e^x-1}{x} < 1+\frac{x}{2}+\frac{x^2}{3!}e^x$ or $e^x < 1+x+\frac{x^2}{2!}+\frac{x^3}{3!}e^x$, and more generally

(2) $$e^x < 1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots+\frac{x^n}{n!}e^x.$$

The situation is somewhat simpler for $x<0$. It follows from $e^x > 1+x$ that $\frac{e^x-1}{x} > 1+\frac{x}{2}$, but now, since $x<0$, $e^x < 1+x+\frac{x^2}{2!}$. The next result of finding means gives us $\frac{e^x-1}{x} < 1+\frac{x}{2}+\frac{x^2}{3!}$ or $e^x > 1+x+\frac{x^2}{2!}+\frac{x^3}{3!}$, etc., and in general

(3) $$e^x > 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^{2\nu-1}}{(2\nu-1)!}$$
and
(4) $$e^x < 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^{2\nu}}{(2\nu)!}.$$

It follows from (1), (2), (3) and (4) that when $x > 0,\ e^x$ lies between $1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^n}{n!}$ and $1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^n}{n!}e^x$, and when $x < 0,\ e^x$ lies between $1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^n}{n!}$ and $1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^n}{n!} + \frac{x^{n+1}}{(n+1)!}$. Thus the error in approximating e^x by $1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^n}{n!}$ is less than $\frac{x^n}{n!}(e^x - 1)$ if $x > 0$, and $\left|\frac{x^{n+1}}{(n+1)!}\right|$ if $x < 0$. But for a fixed value of x, $\lim_{n\to\infty} \frac{x^n}{n!} = 0$. [By 10., each of the products $2 \cdot (n-1), 3 \cdot (n-2), \ldots, (n-1) \cdot 2$ is greater than $1 \cdot n$. Thus $(n-1)!^2 > n^{n-2},\ n!^2 > n^n$ (for $n \geq 3$) and $n! > \sqrt{n^n}$. It follows that $\left|\frac{x^n}{n!}\right| < \left|\frac{x}{\sqrt{n}}\right|^n$. If $n > 4x^2$ so that $\sqrt{n} > |2x|$, then $\left|\frac{x^n}{n!}\right| < \left|\frac{x}{\sqrt{n}}\right|^n < \left(\frac{1}{2}\right)^n$, and therefore $\lim_{n\to\infty} \frac{x^n}{n!} = 0$.]

Thus
(5) $$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$$
for all real numbers x.

Note 1. This series is particularly well suited for computing e :
$e = 1 + \frac{1}{1!} + \frac{1}{2!} + \ldots + \frac{1}{10!} = 2.71828180115$ with an error of

$$\begin{aligned} \frac{1}{11!} + \frac{1}{12!} + \frac{1}{13!} \ldots &= \frac{1}{11!}\left(1 + \frac{1}{12} + \frac{1}{12 \cdot 13} + \ldots\right) \\ &< \frac{1}{11!}\left(1 + \frac{1}{12} + \frac{1}{12^2} + \frac{1}{12^3} + \ldots\right) \\ &= \frac{1}{11!} \cdot \frac{12}{11} \\ &= 2.73295727841 \times 10^{-8} \end{aligned}$$

Note 2. Dörrie continues with a discussion of extending the exponential function to the complex numbers by the definition

$$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \ldots$$

Note 3. Dörrie shows that the series converges absolutely for all complex numbers z, and that $e^a \cdot e^b = e^{a+b}$ for any two complex numbers a and b.

Note 4. Thus if $z = x + iy$, x and y real and $i = \sqrt{-1}$, $e^z = e^x \cdot e^{iy}$, and since

$$
\begin{aligned}
e^{iy} &= 1 + iy - \tfrac{y^2}{2!} - i\tfrac{y^3}{3!} + \tfrac{y^4}{4!} + i\tfrac{y^5}{5!} - \tfrac{y^6}{6!} - + + - - \dots \\
&= \left(1 - \tfrac{y^2}{2!} + \tfrac{y^4}{4!} - \tfrac{y^6}{6!} + - \dots\right) + i\left(y - \tfrac{y^3}{3!} + \tfrac{y^5}{5!} - \tfrac{y^7}{7!} + - \dots\right) \\
&= \cos y + i \sin y \quad \text{by No. 15}
\end{aligned}
$$

$$e^z = e^x(\cos y + i\sin y).$$

Note 5. Let $y = \pi$ in $e^{iy} = \cos y + i\sin y$ to get $e^{i\pi} = -1,$ Euler's formula.

Note 6. Addition and subtraction of $e^{iy} = \cos y + i\sin y$ and $e^{-iy} = \cos(-y) - i\sin(-y) = \cos y - i\sin y$ give the remarkable pair of formulas:

$$\cos y = \frac{e^{iy} + e^{-iy}}{2} \text{ and } \sin y = \frac{e^{iy} - e^{-iy}}{2i}.$$