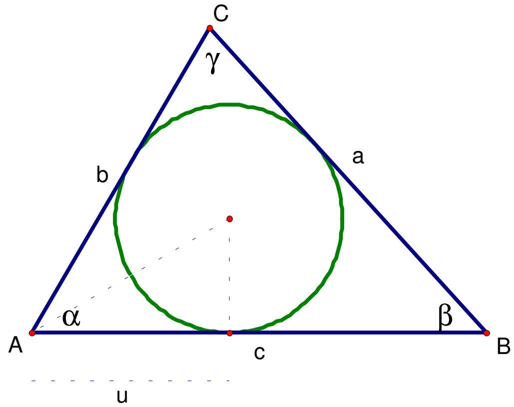
17. Gregory's ArcTangent Series

Determine the angles of a triangle from the sides without the use of tables (or a calculator with inverse trig functions).

We use the following notation



together with

Radius of the incircle	ρ		
Semiperimeter	s	$=$	$\frac{a+b+c}{2}$
	u	$=$	$s-a$
	v	$=$	$s-b$
	w	$=$	$s-c$

These are all easily found in terms of the sides. This is clear for s, u, v and w, but also for ρ, since the area of the triangle is $s\rho$, and this (using Heron's area formula) leads to $\rho^2 = \frac{uvw}{s}$. In addition, the incenter lies on the angle bisectors of each angle, so

$$\tan\frac{\alpha}{2} = \frac{\rho}{u}, \qquad \tan\frac{\beta}{2} = \frac{\rho}{v}, \qquad \tan\frac{\gamma}{2} = \frac{\rho}{w},$$

and it suffices to determine $\tan^{-1}x$. [This is routinely done in a Calculus course nowadays, but $\tan^{-1}x$ can also be found using the inverse $\tan$ function on a calculator without knowing any Calculus.]

The task is to find a power series for $\tan^{-1}(x)$, a problem solved by the English mathematician James Gregory (1638-1675) in 1671. We start with the average value of the function $\frac{1}{1+x^2}$ on the interval $[0,x]$. Since the average value of $f(x)$ on $[a,b]$ (from a Calculus course) is $\frac{1}{b-a}\int_a^b f(t)dt$, the average value in question is $\frac{1}{x}\int_0^x \frac{1}{1+t^2}dt = \frac{\tan^{-1}x}{x}$. (We also assume that the reader also knows that $\frac{d}{dt}\tan^{-1}t = \frac{1}{t^2+1}$.)

Let $F(x) = \frac{1}{1+x^2}$. For brevity's sake, write F for $F(x)$. Then

$$\begin{aligned} F &= 1 - x^2F \\ &= 1 - x^2(1 - x^2F) \qquad = 1 - x^2 + x^4F \\ &= 1 - x^2 + x^4 - x^6F \\ &= 1 - x^2 + x^4 - x^6 + x^8F \\ &= 1 - x^2 + x^4 - x^6 + x^8 - x^{10}F \end{aligned}$$

etc.,

and we obtain the inequality

$$1 - x^2 + x^4 - + \ldots - x^{4n-2} \le F \le 1 - x^2 + x^4 - + \ldots - x^{4n-2} + x^{4n}.$$

[Equality holds at $x = 0$.] Note that

1. The average value of t^n on $[0,x]$ is $\frac{1}{x}\int_0^x t^n dt = \frac{x^n}{n+1}$, and

2. If $f(t) \le g(t)$ on $[a,b]$, then the same is true for their average values.

Thus

$$1 - \frac{x^2}{3} + \frac{x^4}{5} - + \ldots - \frac{x^{4n-2}}{4n-1} \le \frac{\tan^{-1}x}{x} \le 1 - \frac{x^2}{3} + \frac{x^4}{5} - + \ldots - \frac{x^{4n-2}}{4n-1} + \frac{x^{4n}}{4n+1}$$

for $x \ne 0$, or

(3) $\qquad x - \frac{x^3}{3} + \frac{x^5}{5} - + \ldots - \frac{x^{4n-1}}{4n-1} \le \tan^{-1}x \le x - \frac{x^3}{3} + \frac{x^5}{5} - + \ldots - \frac{x^{4n-1}}{4n-1} + \frac{x^{4n+1}}{4n+1}$

for all real x. The error in approximating $\tan^{-1}x$ by the left or right hand sum is no larger than $\frac{x^{4n+1}}{4n+1}$ but for $|x| < 1$, this tends toward zero, and thus when $|x| < 1$ [also when $x = 1$], we obtain the infinite series

(4) $$\tan^{-1}x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + - \ldots$$

This is *Gregory's formula. If the series is terminated at some point, the error incurred is smaller than the first disregarded term.*

The series cannot be used when $|x| > 1$, because it no longer converges. In order to calculate $\tan^{-1}x$ in this case, we introduce $y = \frac{1}{x}$ and use the formula

(5) $$\tan^{-1}x + \tan^{-1}y = \frac{\pi}{2},$$

and with $\alpha = \tan^{-1}x$, we get $y = \tan\left(\frac{\pi}{2} - \alpha\right) = \frac{1}{\tan\alpha}$, and $\tan^{-1}y = \frac{\pi}{2} - \tan^{-1}x$ or

$$\tan^{-1}x + \tan^{-1}\frac{1}{x} = \frac{\pi}{2}.$$

(4) may not be advisable to use when x is very close to 1. In this case let $z = \frac{1-x}{1+x}$, and

use the result

(6) $$\tan^{-1}x + \tan^{-1}z = \tfrac{\pi}{4}.$$

[This follows from $\tan\left(\frac{\pi}{4} - \alpha\right) = \frac{1-\tan\alpha}{1+\tan\alpha}$ or $\frac{\pi}{4} - \alpha = \tan\frac{1-x}{1+x} = \tan^{-1}z$.] Thus we can obtain $\tan^{-1}x$ from $\tan^{-1}z$ (and $\frac{\pi}{4}$).

Note. If we set $x = 1$ in (4), we get Leibniz's series:

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + - \ldots$$

which was discovered by Leibniz in 1674 independently of Gregory.

It is not advisable, however to use this series to approximate π. A series, discovered by the English mathematician John Machin (†1751) and published by him in 1706, is much better suited for this purpose. Machin used an auxiliary angle λ whose tangent is $\frac{1}{5}$. From $\tan\lambda = \frac{1}{5}$, it follows that $\tan 2\lambda = \frac{2\tan\lambda}{1-\tan^2\lambda} = \frac{5}{12}$, and from this in a similar fashion, $\tan 4\lambda = \frac{120}{119}$. Then $\tan^{-1}\frac{120}{119} = 4\tan^{-1}\frac{1}{5}$, or on using (5) and (6), $\frac{\pi}{4} + \tan^{-1}\frac{1}{239} = 4\tan^{-1}\frac{1}{5}$. Thus

$$\frac{\pi}{4} = 4\tan^{-1}\frac{1}{5} - \tan^{-1}\frac{1}{239}$$

or written out more fully:

$$\frac{\pi}{4} = 4\left(\frac{1}{5} - \frac{1}{3 \cdot 5^3} + \frac{1}{5 \cdot 5^5} - + \ldots\right) - \left(\frac{1}{239} - \frac{1}{3 \cdot 239^3} + \frac{1}{5 \cdot 239^5} - + \ldots\right)$$

Machin used this series to calculate π to 100 decimal places.