23. Gauss’ Fundamental Theorem of Algebra

Every polynomial equation of the \( n \)th degree
\[
z^n + c_1z^{n-1} + c_2z^{n-2} + \ldots + c_n = 0
\]
has \( n \) roots, more precisely: every polynomial
\[
f(z) = z^n + c_1z^{n-1} + c_2z^{n-2} + \ldots + c_n
\]
can be factored into a product of \( n \) linear factors of the form \( z - \alpha_i \).

This famous theorem, the fundamental theorem of algebra, was first stated by d’Alembert in 1746, but only partially proved. The first rigorous proof was given in 1799 by the twenty-one year old Gauss in his doctoral dissertation *Demonstratio nova theorematis omnem functionem algebraicam rationalem integrum unius variabilis in factores reales primi vel secundi gradus resolvi posso* (Helmstaedt, 1799). Gauss subsequently gave three other proofs of this theorem. All four are found in the third volume of his *Works*. Other authors after Gauss, including Argand, Cauchy, Ullherr, Weierstrass and Kronecker also gave proofs of the fundamental theorem. The following proof (as modified by Cauchy) is Argand’s (*Annales de Gergonne*, 1815), which is noted for its brevity and simplicity.

This proof (like most of the other ones) has two steps. The first - and more difficult - step proves that a polynomial equation of the \( n \)th degree always has at least one root; the second step shows that it has exactly \( n \) roots.

**First Step**

We set
\[
z^n + c_1z^{n-1} + c_2z^{n-2} + \ldots + c_n = f(z) = w
\]
and consider the different values that are assumed by \( |w| \) when for any complex number \( z \), \( |w| \geq 0 \), and we have the possibilities that \( w \) has a minimum \( \mu > 0 \) or a minimum \( 0 \) at some number \( z_0 \). [Why does \( f \) have a minimum value? We can write \( f(z) = z^n \left[ 1 + \frac{c_1}{z} + \ldots + \frac{c_n}{z^n} \right] \) and upon setting \( |z| = r \), we have \( |f(z)| \geq r^n \left[ 1 - \frac{|c_1|}{r} - \ldots - \frac{|c_n|}{r^n} \right] \), which can be made arbitrarily large for sufficiently large \( |z| \). Say \( |f(z)| > |c_n| \) for \( |z| > R \). The real valued continuous function \( |f(z)| \) has a minimum \( \mu \) (and a maximum) on the compact set \( |z| \leq R \), and this minimum value is less than or equal to \( |f(0)| = |c_n| \). It follows that \( |f(z)| \geq \mu \) for all complex \( z \), with equality holding at some point \( z_0 \).]

Suppose that \( \mu > 0 \). In every circle \( K = z_0 + re^{i\theta} \) centered at \( z_0 \), \( |w| \geq \mu \), since \( \mu \) is the minimum of \( |w| \).
For any \( z \) inside \( K, z = z_0 + \rho e^{i\theta} \) for some \( \rho < r \), and some \( \theta \) the angle of inclination of segment \( z_0z \) from the real axis. Let \( \zeta = \rho e^{i\theta} \). Then

\[
\begin{align*}
w = f(z) &= f(z_0 + \zeta) = (z_0 + \zeta)^n + c_1(z_0 + \zeta)^{n-1} + \ldots + c_n \\
&= z_0^n + c_1z_0^{n-1} + c_2z_0^{n-2} + \ldots + c_n + \frac{d_1\zeta}{1} + \frac{d_2\zeta^2}{2} + \ldots + d_n \zeta^n \\
&= f(z_0) + d_1 \zeta + d_2 \zeta^2 + \ldots + d_n \zeta^n
\end{align*}
\]

Let \( d \) be the first nonzero of the \( d_i \)s, \( d' \) the next, etc. Then we can write
\( w = w_0 + d \zeta v + d' \zeta v' + d'' \zeta v'' + \ldots \) with \( v < v' < v'' < \ldots \) and \( w_0 = f(z_0) \). Divide by \( w_0 \) and factor to get
\[
\frac{w}{w_0} = 1 + q \zeta v (1 + \zeta \xi),
\]

where \( q = \frac{d}{w_0} \) and \( \xi \) is a sum of different positive powers of \( \zeta \).

We now consider the product \( q \zeta v (1 + \zeta \xi) \). Let \( q = he^{i\beta} \). Then \( q \zeta v = hpe^{i(\beta + v \vartheta)} \). From now on, we will consider only \( z \) inside \( K \) for which \( \lambda + v \vartheta = \pi \), i.e., those for which \( \vartheta = \frac{\pi - \lambda}{v} \).

For all such \( z \), \( q \zeta v (1 + \zeta \xi) = hp e^{i\beta} (1 + \zeta \xi) = -hp v (1 + \zeta \xi) \). Since \( \rho = |\zeta| < R \), we can make \( 1 + \zeta \xi \) as close to \( 1 \) as desired by making \( R \) small enough, and thus
\[
\frac{w}{w_0} = 1 - hp v (1 + \zeta \xi)
\]

as close to \( 1 - hp v \) as desired. In particular, there will be \( z \)s in \( K \) for which \( \left| \frac{w}{w_0} \right| < 1 \), i.e., for which \( lw1 < lw01 = \mu \), the minimum value of \( lw1 \). This contradiction shows that \( \mu > 0 \) is untenable, and thus \( \mu = 0 \), i.e., every polynomial equation has at least one root.

Second Step

If \( f(z) \) is a polynomial of degree \( n \) with a root \( \alpha_1 \), then the division algorithm shows that
\[
f(z) = (z - \alpha_1)f_1(z) + R
\]

where \( f_1(z) \) is a polynomial of degree \( n - 1 \), and \( R \) is a constant. If fact \( R = 0 \) as can be seen by setting \( z = \alpha_1 \). Thus every polynomial in \( z \) can be represented as a product of a linear factor \( z - \alpha_1 \) with a polynomial of one degree lower.

\( f_1(z) \), as long as it’s not a constant polynomial, has a root by the First Step, and we can write
\[ f_1(z) = (z - \alpha_2)f_2(z). \]

Doing this \( n \) times (starting with \( f(z) \)) results in
\[ f(z) = (z - \alpha_1)(z - \alpha_2) \ldots (z - \alpha_n). \]

Thus every polynomial equation of the \( n^{th} \) degree can be written as the product of \( n \) linear factors. It follows that \( f(z) = 0 \) has the \( n \) roots \( \alpha_1, \alpha_2, \ldots, \alpha_n \) and no others.

**Note.** It is possible that several of the \( n \) roots \( \alpha_1, \alpha_2, \ldots, \alpha_n \) are equal, for example \( \alpha_1 = \alpha_2 = \alpha_3 \) while all the others are different from \( \alpha_1 \). In this case \( \alpha_1 \) is called a multiple root, more specifically a triple root or a root of multiplicity three.