30. Malfatti’s Problem

Within a given triangle, construct three circles each of which is tangent to the other two and to two sides of the triangle.

![Diagram of Malfatti's Problem](https://via.placeholder.com/150)

Figure 1

This famous problem was posed by the Italian mathematician Malfatti (1731-1807) in 1803 and solved in the tenth volume of the Memorie di Matematica e di Fisica della Società Italiana delle Scienze. This algebraic-geometric solution can be found, for example, in vol. 123 of Ostwald's Klassiker der exakten Wissenschaften (Supplement). The purely geometric solution of Malfatti’s problem given by Jacob Steiner from 1826 without proof is also described there and proved there. Here we will restrict ourselves to the exposition of the thoroughly simple solution published by Schellbach in volume 45 of Crelle’s Journal.

Let $\triangle ABC$ be the given triangle with sides $a, b, c$, perimeter $2s$. Let the Malfatti circles we are looking for be $C_P, C_Q, C_R$ with centers $P, Q, R$ and radii $p, q, r$ respectively. Let the tangents from $A, B, C$ to the circles $C_P, C_Q, C_R$ be $u, v, w$.

![Diagram of Malfatti's Problem with tangents and incircle](https://via.placeholder.com/150)

Figure 2

Consider the incircle $C_I$ of $\triangle ABC$ with center $I$ and radius $\rho$. Let $a_1, b_1, c_1$ be the tangents from $A, B, C$ to the points of tangency:
Then

\[ b_1 + c_1 = a, \quad c_1 + a_1 = b, \quad a_1 + b_1 = c, \]

and (since \( s = a_1 + b_1 + c_1 = a_1 + a = b_1 + b = c_1 + c \))

\[ a_1 = s - a, \quad b_1 = s - b, \quad c_1 = s - c. \]

Furthermore the square of the area is

\[ s(s - a)(s - b)(s - c) = sa_1b_1c_1 \]

by Heron’s formula, but also \((ps)^2\), and so \(p^2 = \frac{a_1b_1c_1}{s}\).

Both centers \(I\) and \(P\) lie on the angle bisector of \(\angle CAB\), so by similar triangles, we conclude that

\[ \frac{p}{p} = \frac{u}{a_1} \quad \text{or} \quad p = \frac{p}{a_1}u. \]

Similarly \(q = \frac{p}{b_1}v\).

Refer to Figure 2 above. \(U\) and \(V\) are points of tangency for \(C_P\) and \(C_Q\) with \(AB\), and \(UV = t\). \(PF \perp QV\) and \(\triangle P Q F\) is a right triangle. Thus

\[ PQ^2 = PF^2 + FQ^2 \quad \text{or} \quad (p + q)^2 = t^2 + (p - q)^2 \]

and on expanding and solving for \(t\), we get

\[ UV = t = 2\sqrt{pq}. \]

Using the values for \(p\) and \(q\) found above, this becomes
\[ t = 2 \sqrt{uv} \sqrt{\frac{\rho^2}{a_1b_1}}. \]

But \( \rho^2 = \frac{ab_1c_1}{s} \), and so we get \( UV = t = 2 \sqrt{\frac{c_1}{s}} \sqrt{uv} \). Since side \( AB \) of \( \triangle ABC \) is composed of the three segments \( AU, UV, VB \), we get

\[ u + v + 2 \sqrt{\frac{c_1}{s}} \sqrt{uv} = c. \]

In the same way

\[ v + w + 2 \sqrt{\frac{a_1}{s}} \sqrt{vw} = a \]

and

\[ w + u + 2 \sqrt{\frac{b_1}{s}} \sqrt{wu} = b. \]

Scale \( \triangle ABC \) if necessary so that \( s = 1 \). Then we have the system of equations

\[ \begin{align*}
  v + w + 2 \sqrt{a_1} \sqrt{vw} & = a, \\
  w + u + 2 \sqrt{b_1} \sqrt{wu} & = b, \\
  u + v + 2 \sqrt{c_1} \sqrt{uv} & = c.
\end{align*} \]

Since \( b + c > a \), it follows that \( a + b + c > 2a \) and \( 1 > a \), since \( s = 1 \). Similarly \( 0 < b, c < 1 \). Then of course \( 0 < u, v, w < 1 \), and there are six (unique) acute angles \( \lambda, \mu, \tau, \psi, \phi, \chi \) such that

\[ \sin^2 \lambda = a, \quad \sin^2 \mu = b, \quad \sin^2 \tau = c, \]

\[ \sin^2 \psi = u, \quad \sin^2 \phi = v, \quad \sin^2 \chi = w. \]

Since \( 1 = a + a_1 = b + b_1 + c + c_1 \), \( \cos^2 \lambda = a_1, \cos^2 \mu = b_1, \cos^2 \tau = c_1 \), and (1) takes the form:

\[ \begin{align*}
  \sin^2 \psi + \sin^2 \chi + 2 \sin \psi \sin \chi \cos \lambda & = \sin^2 \lambda, \\
  \sin^2 \chi + \sin^2 \psi + 2 \sin \chi \sin \psi \cos \mu & = \sin^2 \mu, \\
  \sin^2 \psi + \sin^2 \phi + 2 \sin \psi \sin \phi \cos \tau & = \sin^2 \tau.
\end{align*} \]

Recall the generalized law of sines \( \frac{a}{\sin \alpha} = \frac{b}{\sin \beta} = \frac{c}{\sin \gamma} = 2R \), where \( R \) is the radius of the circumcircle. If \( R = \frac{1}{2} \), then \( a = \sin \alpha, b = \sin \beta \) and \( c = \sin \gamma \).
Then \( \sin^2\gamma = \sin^2\alpha + \sin^2\beta - 2\sin\alpha\sin\beta\cos\gamma \) by the law of cosines, or
\[ \sin^2\gamma' = \sin^2\alpha + \sin^2\beta + 2\sin\alpha\sin\beta\cos\gamma' \] where \( \gamma' = 180^\circ - \gamma = \alpha + \beta \). The converse is also true for acute angle \( \alpha, \beta, \gamma' \), i.e. if \( \sin^2\gamma' = \sin^2\alpha + \sin^2\beta + 2\sin\alpha\sin\beta\cos\gamma' \), then \( \gamma' = \alpha + \beta \). Thus (2) becomes \( \psi + \chi = \lambda, \chi + \psi = \mu, \chi + \varphi = \tau \) and from this
\[
\psi = \sigma - \lambda, \quad \varphi = \sigma - \mu, \quad \chi = \sigma - \tau \quad \text{with} \quad \sigma = \frac{\lambda + \mu + \tau}{2}.
\]

The steps in the argument are reversible and lead to \( PQ = p + q \), so the circles constructed from \( \lambda, \mu, \tau, \psi, \varphi, \chi \) are the Malfatti circles of \( \triangle ABC \).

We have the following construction for \( u, v \) and \( w \), the tangents to the Malfatti circles from the vertices, assuming that \( s = 1 \):

1. Construct \( \lambda, \mu, \tau \) so that \( \sin^2\lambda = a, \sin^2\mu = b \) and \( \sin^2\tau = c \). (See Note 1 below.)
2. Construct \( \sigma = \frac{\lambda + \mu + \tau}{2} \), and then \( \psi = \sigma - \lambda, \varphi = \sigma - \mu, \chi = \sigma - \tau \).
3. Construct \( \sin^2\psi = u, \sin^2\varphi = v \), and \( \sin^2\chi = w \). (See Note 1 below.)

**Note 1.** How to construct

1. \( w \) such that \( \sin^2w = m \), given \( m \).
2. \( m = \sin^2w \) given \( w \).

**Consider the following figure** of a semicircle with diameter \( HK = 1 \):
\[ m = HM = HL \sin \angle HLM = HL \sin w \quad \text{and} \quad HL = HK \sin w = \sin w, \quad \text{so} \quad m = \sin^2 w. \]

**Note 2.** Malfatti (and others) thought that this was the way to inscribe three circles with maximal area in a triangle. (Malfatti was trying to solve the problem of cutting three circular columns of largest volume from a triangular block of marble.) Malfatti circles do not always give the maximal area, e.g.

\[ \text{Malfatti area} = 4.44 \text{ cm}^2 \]

\[ \text{Area} = 7.06 \text{ cm}^2 \]

**Note 3.** It’s a good exercise to use some sort of geometry software to build a tool to construct the Malfatti circles for a triangle.