35. The Delian Cube-doubling Problem

To construct the edge of a cube that is double the size (volume) of a given cube.

According to an account by the mathematician and historian Eutocius (sixth century A.D.), the name "Delian problem" goes back to an old legend according to which the Delphic oracle in one of its utterances demanded that the Delian altar block be doubled.

If k is the edge of the given cube and x the edge of the cube we are seeking,satisfies the equation $x^3 = 2k^3$, and $x = \sqrt[3]{2}\,k$. This problem is not solvable with compass and straightedge alone. (See No. 36.)

The numerous solutions to this problem, some from antiquity, consequently use other means. The solution of the Greek mathematician Menaechmus (ca. 375-325 B.C.) is based on finding the point ($\neq (0,0)$) of intersection of the two parabolas:

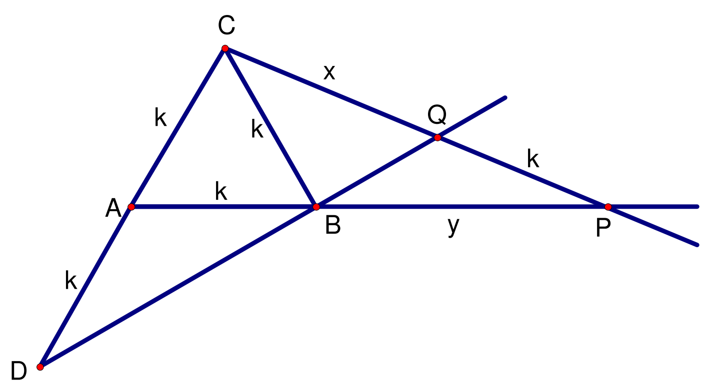
(1) $x^2 = ky$ and

(2) $y^2 = 2kx$,

at which $x^4 = k^2y^2 = k^2 2kx = 2k^3x$, and $x^3 = 2k^3$. Descartes (1596-1650) showed that one of the two parabolas was sufficient. Any points of intersection of the two satisfy the equation $x^2 + y^2 = ky + 2kx$, an equation of a circle with center $\left(k, \frac{k}{2}\right)$ passing through the common vertex of both parabolas. Thus, we need only find the intersection of this circle with one of the two parabolas to find the desired point of intersection.

The simplest way to "construct" $x = k\sqrt[3]{2}$ is by *paper strip construction*:

1. Construct an equilateral triangle $\triangle ABC$ with side k, extend CA by $AD = k$ and draw line DB.
2. Mark the distance k on the sharp edge of a paper strip.
3. Place the paper strip in such a way that the edge passes through C and the end points of the marked-off distance fall upon two points P and Q of the extensions of AB and DB :



Then $CQ = x = k\sqrt[3]{2}$.

Proof. As shown in the figure above, $CQ = x$ and $BP = y$. Apply the law of cosines to $\triangle APB$ to conclude that

$$\begin{aligned} CP^2 &= AC^2 + AP^2 - 2 \cdot AC \cdot AP \cdot \cos 60^\circ \\ &= AC^2 + AP^2 - AC \cdot AP \\ CP^2 - AC^2 &= AP(AP - AC) \\ &= AP(AP - AB) \\ &= AP \cdot BP = BP \cdot AP \\ (x+k)^2 - k^2 &= y(k+y) \text{ or} \\ x^2 + 2kx &= y^2 + ky. \end{aligned}$$

Now apply Menelaus' Theorem to $\triangle APC$ with transversal DBQ to conclude that $\frac{AB}{BP} \cdot \frac{PQ}{QC} \cdot \frac{CB}{DA} = 1$, i.e., $\frac{k}{y} \cdot \frac{k}{x} \cdot \frac{2k}{k} = 1$ or $xy = 2k^2$. Thus we have

(I) $\quad x^2 + 2kx = y^2 + ky$ and

(II) $\quad xy = 2k^2$.

It follows that (if $x \neq 0$)

$$\begin{aligned} x^2 + 2kx &= \left(\tfrac{2k^2}{x}\right)^2 + k\left(\tfrac{2k^2}{x}\right) \\ &= \tfrac{4k^4}{x^2} + \tfrac{2k^3}{x} \\ x^4 + 2kx^3 &= 4k^4 + 2k^3x \\ x^3(2k+x) &= 2k^3(2k+x) \\ x^3 &= 2k^3 \end{aligned}$$

and $x = k\sqrt[3]{2}$.

Note 1. The pairs of curves (1) and (2), and (I) and (II) each have two points of intersection, being second degree curves. It is clear that any points of intersection of (1) and (2) are also points of intersection of (I) and (II). Thus the pairs of curves have the same points of intersection. This gives another argument for $x^3 = 2k^3$.

Note 2. Nicomedes (ca. 240 B.C.) devised a similar "paper strip" construction for doubling the cube, and invented an apparatus to construct the curve, the so-called *conchoid*, that determines how to mark off the distance k on a suitable line segment. See for example The **History of Mathematics** by Burton.

Note 3. Isaac Newton (1642-1727) also devised a "paper strip" construction for doubling the cube. See for example The **History of Mathematics** by Burton.