37. The Regular Heptadecagon

To construct a regular heptadecagon, i.e., a regular 17-gon.

This celebrated problem was solved by Gauss in his major work *Disquisitiones arithmeticae*, published in 1801. In the section dealing with solutions to $x^n = 1$, Gauss proved the following

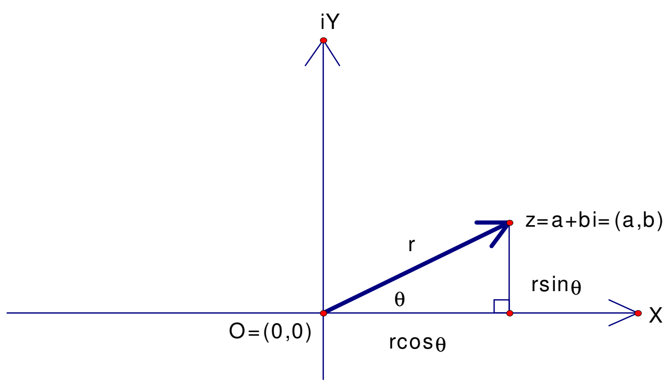
Theorem A regular n-gon can be constructed with compass and straightedge if and only if $n = 2^m p_1 p_2 \ldots p_v$, where $p_1, p_2, \ldots, p_v$ are distinct prime numbers of the form $2^k + 1$.

For $m = 0, v = 1$ and $p_1 = 3$ and $p_1 = 5$ we get the cases of the equilateral triangle and pentagon respectively, which had already been solved in antiquity. [$m = 0, v = 2, p_1 = 3$ and $p_2 = 5$ gives the case of the regular 15-gon, also found in Euclid.] Gauss said "The division of a circle into three and into five equal parts was already known in Euclid's time; it is amazing that nothing new was added to these discoveries in the next two thousand years, that geometers (mathematicians) considered it as settled that, except for these cases and those that could be derived from them, regular polygons could not be constructed with compass and straightedge."

Gauss' advances were possible because he transformed the originally purely geometrical problem into an algebraic one. He arrived at this transformation in the course of representing complex numbers in the plane. An arbitrary complex number $z = a + bi$ is usually represented the point (a, b) in the plane; this point (or the vector from O to (a, b)) itself is call "the complex number z." Another common representation is the trigonometric form

$$z = r(\cos\theta + i\sin\theta)$$

where $r = |z|$ is the magnitude or modulus of z, its distance from the origin O, and θ is the angle or argument of z, θ being measured from the positive real axis to z (thought of as a vector.)



The points on the unit circle $|z| = 1$ are of the form

$$e^{i\varphi} = \cos\varphi + i\sin\varphi.$$

$(e^{i\varphi})^n = e^{i(n\varphi)}$ or

$$(\cos\varphi + i\sin\varphi)^n = \cos n\varphi + i\sin n\varphi,$$

which is known as Demoivre's formula (Abraham Demoivre, 1667-1754).

To obtain a regular polygon of n sides we mark off the angle $\varphi = \frac{2\pi}{n}$ n times in succession from 1 on the unit circle. The resulting points

$$\begin{aligned} \varepsilon_1 = \varepsilon &= \cos\varphi + i\sin\varphi \\ \varepsilon_2 &= \cos 2\varphi + i\sin 2\varphi \\ &\vdots \\ \varepsilon_n &= \cos n\varphi + i\sin n\varphi = 1 \end{aligned}$$

$\varepsilon_\nu = \varepsilon_1^\nu = \varepsilon^\nu$ and $\varepsilon_\nu^n = \varepsilon^{\nu n} = (\varepsilon^n)^\nu = (\cos 2\pi + i\sin 2\pi)^\nu = 1,$ and the n points $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n$ of a regular n-gon are therefore roots of the equation $z^n = 1.$ Thus the geometric problem of "constructing a regular n-gon" by Gauss, is equivalent to the problem "of finding (constructing) the roots of the equation $z^n = 1.$"

The roots of $z^n = 1,$ aside from $z = 1,$ satisfy the equation

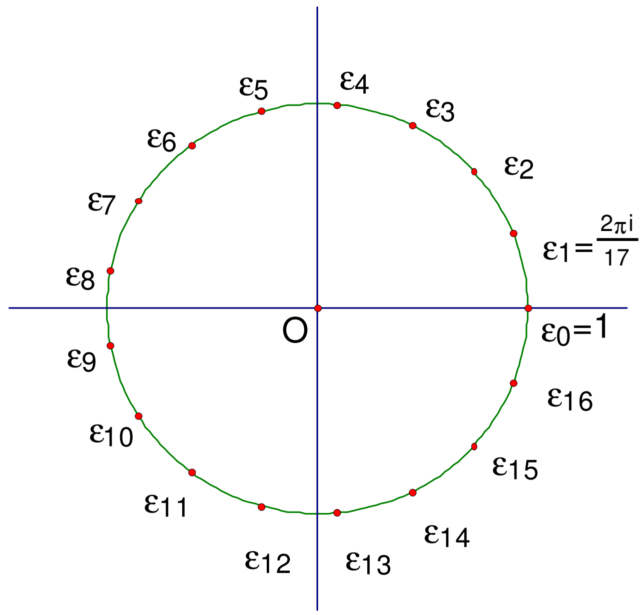
$$\frac{z^n - 1}{z - 1} = z^{n-1} + z^{n-2} + \ldots + z^2 + z + 1 = 0,$$

the n^{th} cyclotomic polynomial. (Dörrie continues with a discussion of how to prove Gauss' Theorem above, without going into all the details. For the sequel, note that the sum of the roots is -1. He then applies this method to the regular 17-gon.)

We will now use Gauss' method to solve the equation, i.e., construct the roots of

$$z^{16} + z^{15} + \ldots + z^2 + z + 1 = 0.$$

Let $\varphi = \frac{2\pi}{17}, \varepsilon = \varepsilon_1 = \cos\varphi + i\sin\varphi, \varepsilon_\nu = \varepsilon^\nu,$ and accordingly $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{17}$ are the vertices of the 17-gon.



Let $g = 3$. g is a primitive root $\operatorname{mod} 17$, i.e., $g^{16} \equiv 1 \operatorname{mod} 17$, but now smaller positive power of g has this property. Thus $\operatorname{mod} 17$, the powers of g, i.e., $g, g^2, g^3, \ldots, g^{16}$ are $1, 2, 3, \ldots, 16$ in some order $(3, 9, 10, \ldots, 1)$, and the roots of the cyclotomic polynomial above are $z_v = \varepsilon^{g^v}, v = 1, 2, \ldots, 16$. Since $z_{v+1} = \varepsilon^{g^{v+1}} = \varepsilon^{g^v g} = z_v^g$, each root in the sequence $z_0, z_1, z_2, \ldots, z_{15}$ is the cube of the preceding one. (Remember $g = 3$.)

$$
\begin{array}{llll}
z_0 = \varepsilon, & z_1 = \varepsilon^3, & z_2 = \varepsilon^9, & z_3 = \varepsilon^{10}, \\
z_4 = \varepsilon^{13}, & z_5 = \varepsilon^5, & z_6 = \varepsilon^{15}, & z_7 = \varepsilon^{11}, \\
z_8 = \varepsilon^{16}, & z_9 = \varepsilon^{14}, & z_{10} = \varepsilon^8, & z_{11} = \varepsilon^7, \\
z_{12} = \varepsilon^4, & z_{13} = \varepsilon^{12}, & z_{14} = \varepsilon^2, & z_{15} = \varepsilon^6.
\end{array}
$$

Let

$$
\begin{aligned}
X &= z_0 + z_2 + z_4 + z_6 + z_8 + z_{10} + z_{12} + z_{14} \\
&= \varepsilon + \varepsilon^9 + \varepsilon^{13} + \varepsilon^{15} + \varepsilon^{16} + \varepsilon^8 + \varepsilon^4 + \varepsilon^2
\end{aligned}
$$

and

$$
\begin{aligned}
x &= z_1 + z_3 + z_5 + z_7 + z_9 + z_{11} + z_{13} + z_{15} \\
&= \varepsilon^3 + \varepsilon^{10} + \varepsilon^5 + \varepsilon^{11} + \varepsilon^{14} + \varepsilon^7 + \varepsilon^{12} + \varepsilon^6.
\end{aligned}
$$

($g = 3$ enters the picture in determining the z_is, and the εs above: each is the $9^{\text{th}} = (g^2)^{\text{th}}$ power of the preceding term.) The sum of the roots of the cyclotomic polynomial is -1, so $X + x = -1$. It is somewhat tedious, but easy to check that Xx equals four times the sum of the roots, so $Xx = -4$. Thus X and x satisfy the quadratic equation

(I) $$t^2 + t - 4 = 0$$

and

$$X = \frac{-1 + \sqrt{17}}{2} \text{ and } x = \frac{-1 - \sqrt{17}}{2}.$$

To see that $X > x$, we note that $\operatorname{Re}\varepsilon^{\mu} = \operatorname{Re}\varepsilon^{\nu}$ if $\mu + \nu = 17$. (See the unit circle above; ε^{μ} and ε^{ν} are conjugates in this case.) From this we obtain

$$\begin{aligned} \operatorname{Re}X &= 2(\operatorname{Re}\varepsilon_1 + \operatorname{Re}\varepsilon_2 + \operatorname{Re}\varepsilon_4 + \operatorname{Re}\varepsilon_8) \quad > 0 \\ \operatorname{Re}x &= 2(\operatorname{Re}\varepsilon_3 + \operatorname{Re}\varepsilon_5 + \operatorname{Re}\varepsilon_6 + \operatorname{Re}\varepsilon_7) \quad < 0. \end{aligned}$$

Next let

$$\begin{aligned} U &= z_0 + z_4 + z_8 + z_{12} &&= \varepsilon + \varepsilon^{13} + \varepsilon^{16} + \varepsilon^4 \\ u &= z_2 + z_6 + z_{10} + z_{14} &&= \varepsilon^9 + \varepsilon^{15} + \varepsilon^8 + \varepsilon^2 \\ V &= z_1 + z_5 + z_9 + z_{13} &&= \varepsilon^3 + \varepsilon^5 + \varepsilon^{14} + \varepsilon^{12} \\ v &= z_3 + z_7 + z_{11} + z_{15} &&= \varepsilon^{10} + \varepsilon^{11} + \varepsilon^7 + \varepsilon^6 \end{aligned}$$

($g = 3$ enters the picture in determining the z_is, and the εs above: each is the $81^{\text{st}} = (g^4)^{\text{th}}$ power of the preceding term, and $U + u = X$.) Here we obtain

$$\begin{aligned} U + u = X, &\quad V + v = x \\ Uu = -1, &\quad Vv = -1, \end{aligned}$$

roots of the quadratic equations

(II) $$t^2 - Xt - 1 = 0 \qquad \text{and} \qquad t^2 - xt - 1 = 0$$

respectively. It follows that

$$\begin{aligned} U &= \tfrac{X+\sqrt{X^2+4}}{2}, \quad V = \tfrac{x+\sqrt{x^2+4}}{2}, \\ u &= \tfrac{X-\sqrt{X^2+4}}{2}, \quad v = \tfrac{x+\sqrt{x^2+4}}{2}, \end{aligned}$$

where we conclude that $U > u$ and $V > v$ from

$$\begin{aligned} \operatorname{Re}U = 2(\operatorname{Re}\varepsilon_1 + \operatorname{Re}\varepsilon_4), &\quad \operatorname{Re}V = 2(\operatorname{Re}\varepsilon_3 + \operatorname{Re}\varepsilon_5), \\ \operatorname{Re}u = 2(\operatorname{Re}\varepsilon_2 + \operatorname{Re}\varepsilon_8), &\quad \operatorname{Re}v = 2(\operatorname{Re}\varepsilon_6 + \operatorname{Re}\varepsilon_7). \end{aligned}$$

Finally let

$$\begin{aligned} W &= z_0 + z_8 &= \varepsilon + \varepsilon^{16} \\ w &= z_4 + z_{12} &= \varepsilon^{13} + \varepsilon^{4}. \end{aligned}$$

($g = 3$ enters the picture in determining the z_is, and the εs above: each is the $6561^{\text{st}} = (g^8)^{\text{th}}$ power of the preceding term.) In this case, $W + w = U$ and $Ww = \varepsilon^5 + \varepsilon^{14} + \varepsilon^3 + \varepsilon^{12} = V$. Since $\operatorname{Re} W = 2 \operatorname{Re} \varepsilon_1$ and $\operatorname{Re} w = 2 \operatorname{Re} \varepsilon_4$, $W > w$. W and w are thus roots of the quadratic equation

(III) $$t^2 - Ut + V = 0.$$

The construction of the regular 17-gon then consists of the following four steps:

1. Construct X and x;
2. construct U and V (from X and x);
3. construct W and w (from U and V) according to (III);
4. find W on the real axis; the perpendicular bisector of the segment OW cuts the unit circle in points ε_1 and ε_{16}. All the other vertices are now determined.

Note 1. Gauss was so proud of this discovery that he wanted a regular 17-gon inscribed on his tombstone. This wish was not carried out.

Note 2. Most geometry software programs include rotation tools that enable the user to easily draw regular n-gons.

Note 3. Primes of the form $2^k + 1$ are called Fermat primes. The only ones known today (2010) are $3, 5, 17, 257$ and 65537.