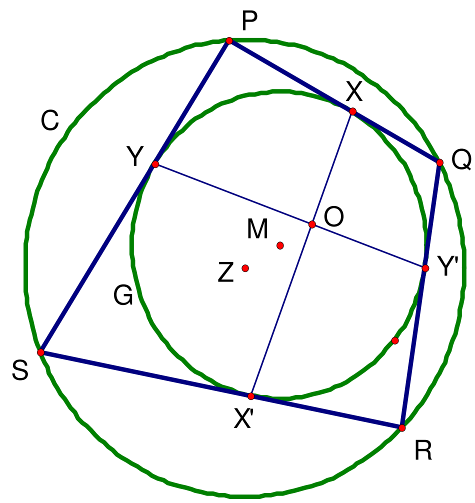
39. Fuss' Problem of the Chord-Tangent Quadrilateral

Find the relation between the radii and the line joining the centers of the circumscribed and inscribed circles of a bicentric quadrilateral.

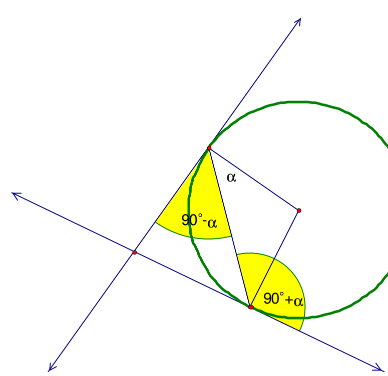
A *bicentric* or *chord-tangent quadrilateral* is one that is simultaneously inscribed in one circle and circumscribed about another. Let $PQRS$ be such a quadrilateral, C its circumcircle and G its inscribed circle.



Let the points of tangency of the opposite sides PQ and RS with circle G be X and X', let the points of tangency of the opposite sides PS and QR with G be Y and Y', and let the O be the point of intersection of chords XX' and YY'. The angle sums in each of the quadrilaterals $OXPY$ and $OX'RY'$ are $360°$ or using a bar to denote the angles (or their measures)

$$\overline{O} + \overline{X} + \overline{P} + \overline{Y} = 360° \text{ and } \overline{O} + \overline{X'} + \overline{R} + \overline{Y'} = 360°.$$

Since $\overline{X} + \overline{X'} = 180° = \overline{Y} + \overline{Y'}$



addition gives the relation:

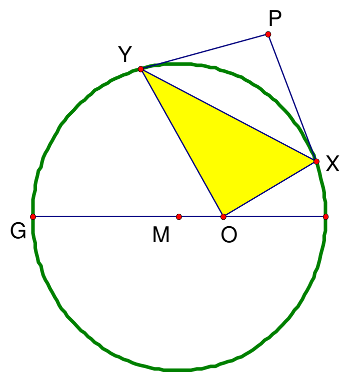
(1) $$2\overline{O} + \overline{P} + \overline{R} = 360°.$$

Since $\overline{P}$ and $\overline{R}$ are opposite angles of the inscribed quadrilateral $PQRS$, $\overline{P} + \overline{R} = 180°$, and thus $O = 90°$, i.e., the chords XX' and YY' are perpendicular.

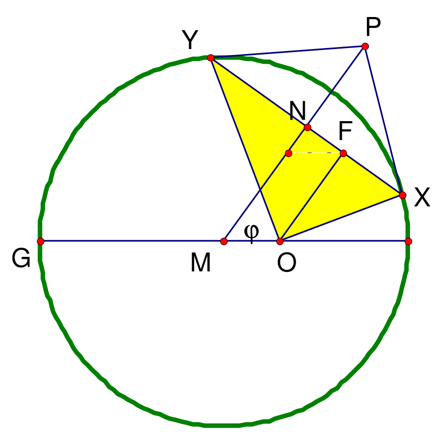
The perpendicularity of XX' and YY' is also sufficient to guarantee that $PQRS$ is bicentric, i.e., $PQRS$ is bicentric if tangents PQ, RS, SP, QR are drawn through the endpoints X, X', Y, Y' of two perpendicular chords XX', YY' of any circle G. Indeed, in this case, $\overline{P} + \overline{R} = 180°$ from (1), and it follows that $PQRS$ is inscribed in a circle.

The simplest way to obtain the desired relation between the radii and the distance between the centers is by means of the following locus problem.

A right angle is rotated about it fixed vertex, which is located inside a circle; find the locus of the point of intersection of the tangents to the circle through the points of intersection of the legs of the angle with the circle, i.e., find the locus of point P as $\overline{O}$ rotates about O in the figure below:



Solution of the locus problem. Let ρ be the radius of circle G, M its center. Let $e = MO$, $p = MP$, and $\varphi = \angle OMP$. Let F be the foot of the altitude in $\triangle OXY$ to XY.



Since $\triangle OXY$ is a right triangle, $OF^2 = FX \cdot FY$. The perpendicular bisector of XY goes through the midpoint N of XY, the center of circle G, and the vertex P os the isosceles triangle $\triangle XYP$. Now let $\rho' = MN$, $e' = e\cos\varphi$, $\rho'' = NX$ and $e'' = e\sin\varphi = NF$. Then $OF^2 = FX \cdot FY$ reads:

$$(\rho' - e')^2 = (\rho'' - e'')(\rho'' + e'') \qquad \text{or}$$
$$2\rho'^2 - 2\rho' e' + e'^2 + e''^2 = \rho'^2 + \rho''^2 \quad \text{or}$$

(2) $$2\rho'^2 - 2\rho' e \cos\varphi + e^2 = \rho^2.$$

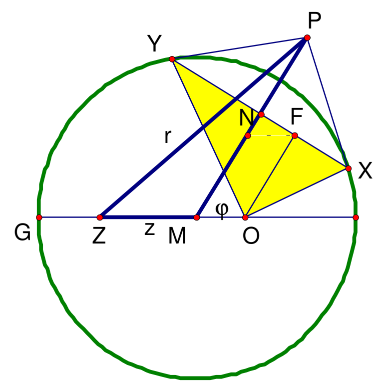
Since right triangles $\triangle MXP$ and $\triangle MNX$ are similar, $\frac{MX}{MP} = \frac{MN}{MX}$, $MX^2 = MP \cdot MN$ or

(3) $$\rho^2 = p\rho'.$$

On substituting ρ' from (3) into (2) and simplifying, we obtain

(4) $$p^2 + 2\frac{\rho^2 e}{\rho^2 - e^2} p \cos\varphi = \frac{2\rho^4}{\rho^2 - e^2}.$$

Now let Z lie on the extension of OM at a distance of $z = ZM$, and let $r = ZP$.



The law of cosines with $\triangle PMZ$ yields

(5) $$r^2 = z^2 + p^2 + 2zp\cos\varphi.$$

Z, up to this point has been any point on the extension of OM. Pick Z now so that

(I) $$MZ = z = \frac{\rho^2 e}{\rho^2 - e^2}.$$

Note that z depends only of the (fixed) circle G and the (fixed) point O, and is thus constant. (5), in light of of (I) and (4) now reads

(II) $$r^2 = z^2 + \frac{2\rho^4}{\rho^2 - e^2},$$

and thus r has a constant value.

It follows that all points P of the locus lie on the circle C with center Z on the extension of OM with M determined by (I) and radius r determined by (II). [Working backwards, we see that every point on C is obtained this way.]

Points Q, R, S, intersection points of tangent lines through the points of intersection of G with the extensions of XO and YO are also on C. Thus $PQRS$ is a bicentric quadrilateral for G and C. All bicentric quadrilaterals for G and C are obtained by rotating $\angle XOY$ about O.

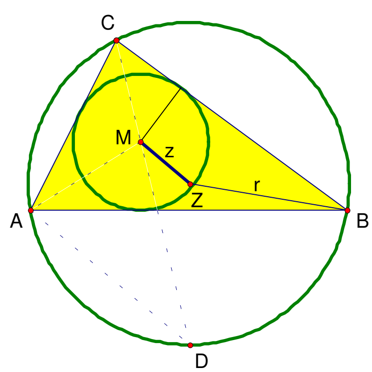
$\frac{1}{\rho^2-e^2} = \frac{r^2-z^2}{2\rho^4}$ from (II), and on substituting into (I) and solving for e, we get $e = \frac{2z\rho^2}{r^2-z^2}$, and from this $\rho^2 - e^2 = \frac{\rho^2\left[(r^2-z^2)^2-4\rho^2z^2\right]}{(r^2-z^2)^2}$. Substitution into (II) (plus some algebra) yields the desired relationship beween the radii r and ρ, and the distance z between the centers:

$$2\rho^2(r^2+z^2) = (r^2-z^2)^2 \quad \text{or}$$

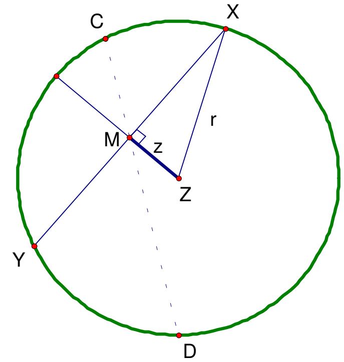
$$\frac{1}{(r-z)^2} + \frac{1}{(r+z)^2} = \frac{1}{\rho^2}$$

This result is from Nicolaus Fuss (1755-1826), a student and friend of Leonhard Euler. Fuss also found the corresponding formulas for bicentric pentagons, hexagons, heptagons and octagons (*Nova Acta Petropol.*, XIII,1798).

Note 1. The corresponding formula for the triangle had already been given by Euler. It reads $r^2 - z^2 = 2r\rho$, where r is the circumradius, ρ the inradius and z the distance between the two centers.



Proof. Let the angles at A, B, C be α, β, γ respectively. Note that D is the midpoint of arc AB because $\angle ACD = \angle BCD = \frac{1}{2}\gamma$. Also $\angle ADM = \angle ADC = \beta$, $\angle DAM = \angle MAB + \angle BAD = \frac{1}{2}\alpha + \angle ACD = \frac{1}{2}\alpha + \frac{1}{2}\gamma$, and $\angle AMD = 180^\circ - \left(\beta + \frac{1}{2}\alpha + \frac{1}{2}\gamma\right) = \frac{1}{2}\alpha + \frac{1}{2}\gamma$, so $\triangle ADM$ is isosceles. Thus $AD = MD$. Next $\sin\frac{\gamma}{2} = \frac{\rho}{MC} = \frac{AD}{2r} = \frac{MD}{2r}$ and $MC \cdot MD = 2r\rho$. Now consider point M relative to the circumcircle:



Then $MC \cdot MD = MX \cdot MY = MX^2 = r^2 - z^2$, and it follows that $r^2 - z^2 = 2r\rho$.

Note 2. From the locus lemma, it follows that a bicentric quadrilateral can always be "rotated" and remain inscribed in C and tangent to G, i.e., from any point on C, draw a tangent of G intersecting C at some point; from that point draw a new tangent to G meeting C in a point, etc. to generate a four-sided Poncelet traverse. This four-sided traverse is closed for any point on C.

Note 3. The French mathematician Poncelet (1788-1867) proved that this theorem is not limited to four-sided traverses. The general result follows. *If an n-sided Poncelet traverse constructed for two conic sections is closed for one starting point, then it is closed for any starting point.*