41. Alhazen’s Billiard Problems

Describe an isosceles triangle whose legs pass through two given points inside a circle.

This problem comes from the Arabic mathematician Abu Ali al Hassan ibn al Hassan ibn Alhaitham (ca. 965-ca.1039), whose name was transformed into Alhazen by the translators of his Optics. In his Optics, the above problem has the following form: "Find the point on a spherical concave mirror at which a ray of light coming from a given point must strike in order to be reflected to another point."

This problem can be posed other forms too, e.g., "There are two balls on a circular billiard table, how must one be struck in order for it to strike the other after rebounding (once) from the cushion?" or "Find the point on a circle the sum of whose distances from two given points within the circle is a minimum (or maximum)."

Many famous mathematicians after Alhazen, among them Huygens, Barrow, L'Hôpital, Riccati and Quètelet also worked on this problem.

Let the given circle be $c = c(O, r)$ with its center at the origin in the $xy$-plane. Let $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ be the given points and $AB_1B_2$ the isosceles triangle that we are looking for with $P_i$ on leg $AB_i$. Because of symmetry $\angle P_1AO = \angle P_2AO$, say $\theta$.

Let $\beta_1, \alpha, \beta_2$ be the angles that lines $P_1A, OA, P_2A$ make with the $x$-axis. Then $\theta = \beta_1 - \alpha$ and $\theta = \alpha - \beta_2$. It follows that
\[
\tan \theta = \frac{\tan \beta_1 - \tan \alpha}{1 + \tan \alpha \tan \beta_1} \quad \text{and} \quad \tan \theta = \frac{\tan \alpha - \tan \beta_2}{1 + \tan \alpha \tan \beta_2}.
\]

But also
\[
\tan \beta_1 = \frac{y_{11}}{x_{11}}, \quad \tan \alpha = \frac{y}{x}, \quad \tan \beta_2 = \frac{y_{21}}{x_{21}},
\]
and hence
\[
\frac{\frac{y_{12}}{x_{12}} - \frac{y}{x}}{1 + \frac{y}{x} \frac{y_{12}}{x_{12}}} = \frac{\frac{y_{22}}{x_{22}} - \frac{y}{x}}{1 + \frac{y}{x} \frac{y_{22}}{x_{22}}}
\]
or
\[
\frac{yx_1 - xy_1}{x^2 + y^2 - xx_1 - yy_1} = \frac{xy_2 - yx_2}{x^2 + y^2 - xx_2 - yy_2}
\]
or on setting
\[
H = x_1y_2 + x_2y_1, \quad K = x_1x_2 - y_1y_2, \quad h = x_1 + x_2, \quad k = y_1 + y_2
\]

\[
H(x^2 - y^2) - 2Kxy + (x^2 + y^2)(hy - kx) = 0.
\]

Since \(A(x, y)\) must lie on circle \(c\), \(x^2 + y^2 = r^2\) and the last condition becomes
\[
H(x^2 - y^2) - 2Kxy + r^2(hy - kx) = 0,
\]
which is an equation of a hyperbola. Thus the desired point \(A(x, y)\) is the point of intersection of

the circle \(x^2 + y^2 = r^2\),

and

the hyperbola \(H(x^2 - y^2) - 2Kxy + r^2(hy - kx) = 0\).

There are in general four solutions, since a circle and hyperbola can intersect in four points.

**Example 1.** Let \(c\) be the circle \(x^2 + y^2 = 5^2\), \(P_1 = (2, 1)\) and \(P_2 = (2, -1)\). Then \(H = 0\), \(K = 5\), \(h = 4\) and \(k = 0\). \((x, y)\) must satisfy \(x^2 + y^2 = 5^2\) and \(-10xy + 100y = 0\), or \(10y(10 - x) = 0\). Since \(|x| \leq 5\), the only solution is \(y = 0\), giving two \(As\): \((\pm 5, 0)\).

**Example 2.** Let \(c\) be the circle \(x^2 + y^2 = 5^2\), \(P_1 = (2, 3)\) and \(P_2 = (2, -3)\). Then \(H = 0\), \(K = 13\), \(h = 4\) and \(k = 0\). \((x, y)\) must satisfy \(x^2 + y^2 = 5^2\) and \(-26xy + 100y = 0\), or \(2y(50 - 13x) = 0\). This gives four \(As\): \((\pm 5, 0), \left(\frac{50}{13}, \pm \frac{5}{13} \sqrt{69}\right)\).
Note 1. Dörrie shows that for $x^2 + y^2 = r^2$, $P_1 = (a, b)$, $P_2 = (a, -b)$ and $a^2 + b^2 > ar$, there are four $A$s. Two are $(\pm r, 0)$. The other $A$s can be found as the intersection points of $x^2 + y^2 = r^2$ and the circumcircle of $\triangle P_1OP_2$.

Note 2. In the case above, Dörrie shows that $A$ (or $A'$) is the point on $x^2 + y^2 = r^2$ the sum of whose distances from $P_1$ and $P_2$ is a minimum. (All that is required in general is that $P_1$ and $P_2$ be equidistant from the center, and after rotating about $O$ if necessary so that $P_1 = (a, b)$ and $P_2 = (a, -b)$, $a^2 + b^2 > ar$.)

Note 3. Dörrie concludes with a discussion of another problem from Alhazen: How should you strike a ball on a circular billiard table so that after striking the cushion twice, the ball will return to its original position? (The solution does not depend on the previous one.)

Solution. Let the billiard table be represented by $x^2 + y^2 \leq r^2$ in the $xy$-plane. Let the initial position of the ball be $P$ (without loss of generality on the positive $x$-axis) at $(c, 0)$. Let the ball first strike the circle at $U$, cross the $x$-axis at a right angle $F$, then strike the circle again at $V$, and finally return to $P$. 
With the notation in the figure above, we have
\[
\frac{y}{z} = \frac{x}{z}, \quad \text{since } \angle PUO = \angle OUF,
\]
\[
r^2 = x^2 + y^2, \quad \text{and}
\]
\[
z^2 = y^2 + (x + c)^2.
\]

Then
\[
z^2 = r^2 - x^2 + (x + c)^2,
\]
\[
(x + y)^2 = r^2 + 2cx + c^2,
\]
\[
c^2y^2 = r^2x^2 + 2cx^3 + c^2x^2,
\]
\[
c^2(r^2 - x^2) = r^2x^2 + 2cx^3 + c^2x^2,
\]
\[
0 = 2cx^3 + r^2x^2 + 2c^2x^2 - c^2r^2,
\]
\[
0 = (2cx^2 + r^2x - cr^2)(x + c),
\]
\[
2cx^2 + r^2x - cr^2 = 0,
\]

and \(x = \frac{-r^2 + r\sqrt{r^2 + 8c^2}}{4c}\) can be constructed from this last equation.