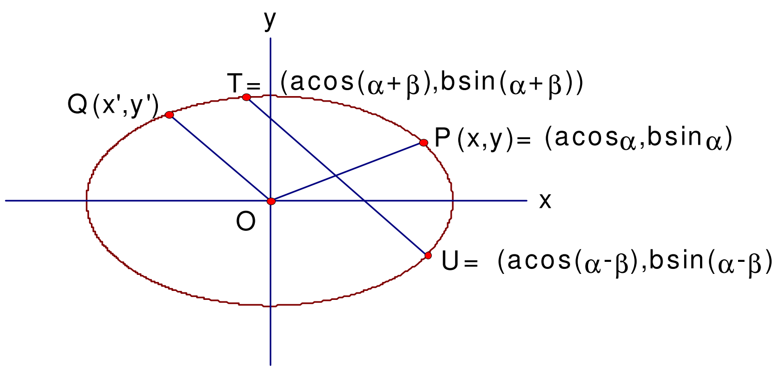
42. An Ellipse from Conjugate Radii

Construct an ellipse given the length and position of two conjugate radii.

Note 1. Preliminary remarks on conjugate radii.

A radius of an ellipse is a segment from the center O of the ellipse to a point on the ellipse. A diameter is a chord of the ellipse that passes through the center. Two diameters are conjugate if each bisects all the chords parallel to the other. Two radii OP and OQ are conjugate if the diameters containing them are conjugate.



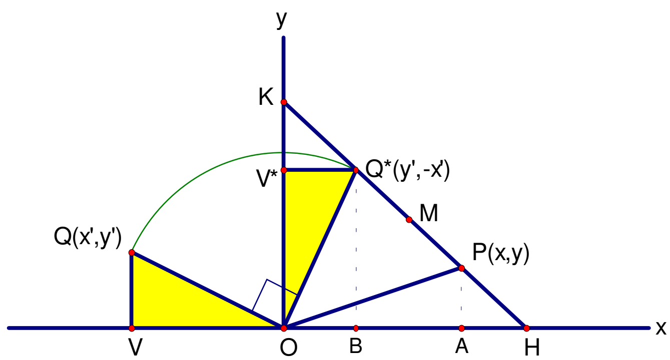
Lines $TU : b\cos\alpha \cdot x + a\sin\alpha \cdot y = ab\cos\beta$ are all parallel for $0^\circ < \beta < 180^\circ$. Midpoints of TU are $(a\cos\alpha\cos\beta, b\sin\alpha\cos\beta)$, all of which lie on line OP, and conversely. Set $\beta = 90^\circ$ to get coordinates of $Q : (x', y') = (-a\sin\alpha, b\cos\alpha)$. Then $x' = \frac{-a}{b}y$ and $y' = \frac{b}{a}x$.

Let the ellipse have the equation

(1) $$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

and the conjugate radii be OP and OQ where $P = (x, y)$ and $Q = (x', y')$. Then

(2) $$\frac{x'}{a} = \frac{-y}{b} \text{ and } \frac{y'}{b} = \frac{x}{a}.$$



Let V be on the x-axis with QV perpendicular to the x-axis. Rotate $\triangle OQV$ clockwise about O through 90° to get $\triangle OQ^*V^*$, and let line PQ^* intersect the axes in points H and K. By (2), $\frac{BQ^*}{AP} = \frac{a}{b}$, and because $\triangle HBQ^* \sim \triangle HAP$, $\frac{HQ^*}{HP} = \frac{a}{b}$ too. Similarly (after projecting Q^* and P onto the y-axis, we see that $\frac{KP}{KQ^*} = \frac{a}{b}$. It then follows that

$$\frac{HP + PQ^*}{HP} = \frac{KQ^* + Q^*P}{KQ^*}, \text{ and thus } HP = KQ^*$$

and thus the midpoint M of PQ^* is also the midpoint of HK. Then

(3) $$\frac{KP}{HP} = \frac{KP}{KQ^*} = \frac{a}{b}.$$

Notice also that $OB = AH$ since $HP = KQ^*$, and $BH = OA = x$. Also $HQ^* = HP + PQ^* = KQ^* + Q^*P = KP$.

In order to get a second equation for the unknowns KP and HP, let $v = \angle KHO$. Then

$$\cos v = \frac{x}{KP} \text{ and } \sin v = \frac{y}{HP};$$

square and add to get

(4) $$\frac{x^2}{KP^2} + \frac{y^2}{HP^2} = 1.$$

It follows directly from (1), (3) and (4) that

$$KP = a \text{ and } HP = b.$$

This gives us the following simple

Construction:

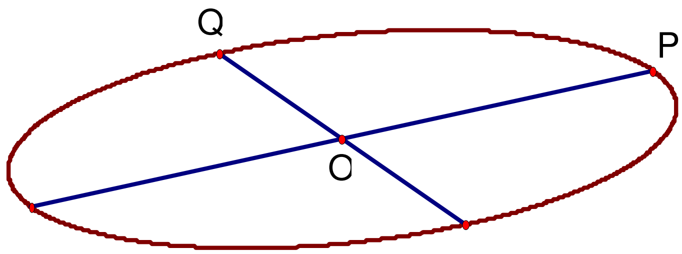
1. Rotate OQ about O 90° through the interior of $\angle POQ$ to OQ^*.
2. Let M be the midpoint of PQ^*.
3. Let H and K be the points of intersection of line PQ^* and the circle center M radius

MO.

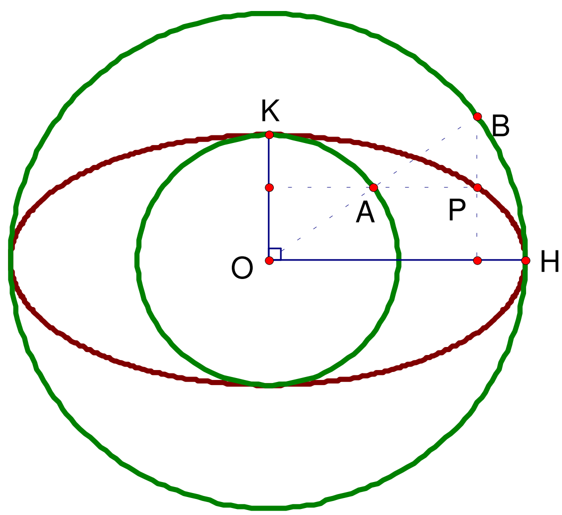
Then

1. *KP and HP are equal to half the length of the axes of the ellipse, and*
2. *OH and OK give the positions of the axes of the ellipse.*

From this, the ellipse can be drawn.



Note 2. To draw an ellipse with axes OH and OK,



1. Construct circles with radii OH and OK,
2. Let A be an arbitrary point on circle OK,
3. B be the point of intersection of ray OA and circle OH,
4. P be the point of intersection of perpendiculars from B to OH and A to OK.
5. Then the locus of P as A moves on circle OK is the ellipse.