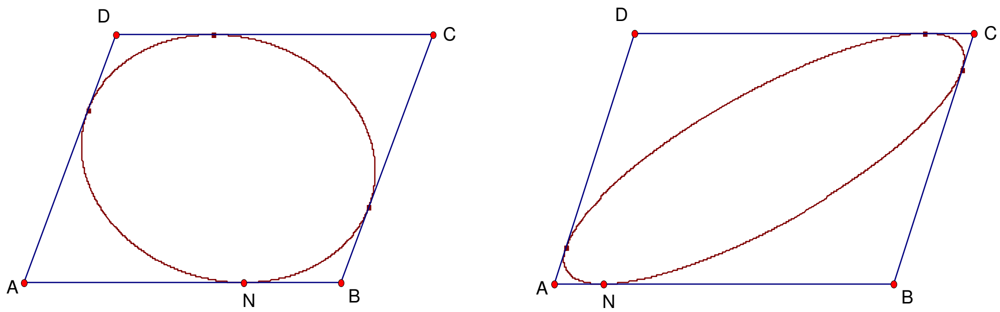
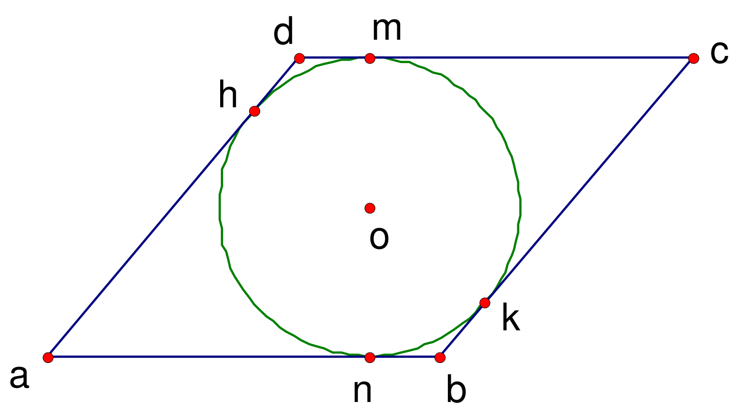
43. An Ellipse in a Parallelogram

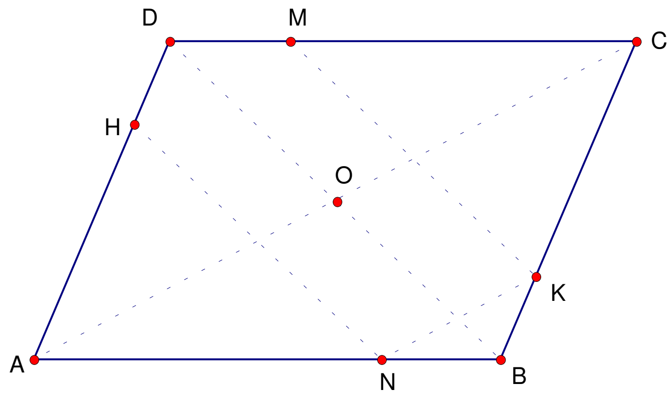
Inscribe an ellipse in a given parallelogram at a specified boundary point.



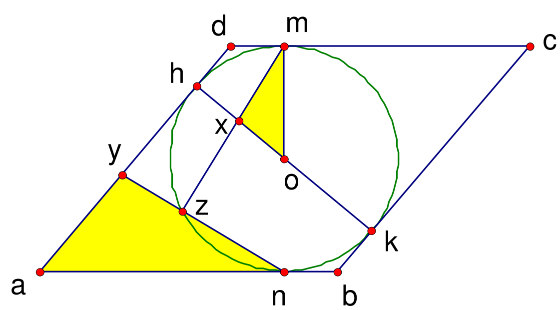
The solution is based on the fact that **every ellipse is the normal projection of a circle.** Let $ABCD$ be the given parallelogram with N the given boundary point on AB. Let the other points at which the ellipse touches the parallelogram be K on BC, M on CD and H on DA. In a normal projection, in which the ellipse is the image of a circle, the parallelogram $ABCD$ and the tangency points N,K,M,H appear as projections of a parallelogram circumscribing a circle, and thus of a rhombus $abcd$ with tangency points n,k,m,h.



Since $nk \| hm \| ac$ and $nh \| km \| bd$ and since normal projection preserves parallelism, $NK \| HM \| AC$ and $NH \| KM \| BD$. It follows that points K,M and H can all be found from N : H and K as the intersection points of parallels through N to BD and AC with DA and BC respectively, and M as the intersection of CD with the parallel through H to AC.

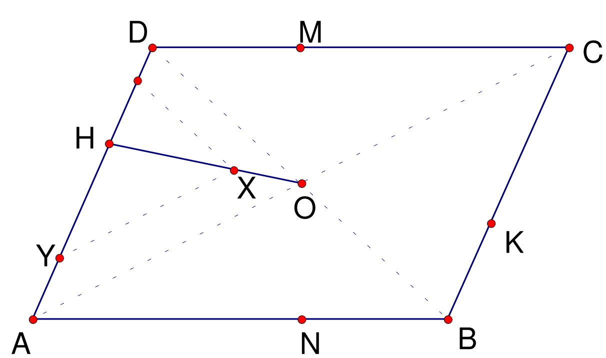


Let the center of the circle be o and the center of the ellipse be O. Let z be any point on arc nh of the circle

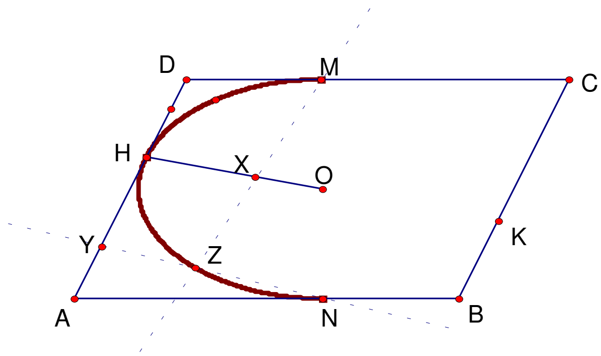


Let x be the intersection of lines zm and hk, and y the intersection of lines zn and ad. Then $\angle xom = 180^\circ - \angle hom = \angle yan$ and $\angle xmo = 90^\circ - \angle znm = \angle yna$ so $\triangle omx \approx \triangle any$. Then $\frac{ox}{om} = \frac{ay}{an}$ and since $om = oh$ and $an = ah$, $\frac{ox}{oh} = \frac{ay}{ah}$. If X, Y, Z are the normal projections of x, y, z, we get $\frac{OX}{OH} = \frac{AY}{AH}$ since normal projection preserves ratios of parallel segments. The points X and Y thus divide OH and AH in the same proportions.

Let X be any point on segment OH and Y (Y') be the intersection of DA and the parallel through X to AC (BD):



so that $\frac{OX}{OH} = \frac{AY}{AH} (= \frac{DY'}{Y'H})$. Then let Z (Z')be the intersection of lines MX (NX)and NY (MY'). Z (Z') is a point on the ellipse. The locus of Z (Z') as X moves on segment OH gives half the ellipse:



Similar constructions produce points on the other half of the ellipse.

Note 1. Use the locus tool of a geometry software program to construct more points on the ellipse.

Note 2. Dörrie's method for finding Z (Z') is the same as this, but it looks somewhat different because of his numbering of points along OH, AH, etc.