44. A Parabola from Four Tangents

Draw a parabola given four lines tangent to it.

The simplest solution of this beautiful problem is based on the following theorem of Lambert (1728-1777), a German mathematician. A "tangent triangle" is a triangle whose vertices are the intersection points of three tangent lines.

**Lambert’s Theorem:** The circumcircle of a tangent triangle of a parabola goes through the focus of the parabola.

![Figure 1](image1.png)

**The proof** follows from the

**Theorem of similar triangles:** With the notation of Figure 1, \( \triangle FSA \approx \triangle FBS \).

![Figure 2](image2.png)

**Proof** Let \( H \) and \( K \) be points on the directrix so that \( AH \) and \( BH \) are perpendicular to the directrix. \( H \) and \( K \) are also mirror images of \( F \) about the tangent lines \( AS \) and \( BS \) respectively.
\( \triangle AFH \) is isosceles, and line \( AS \) is the angle bisector of \( \angle A \) as well as the altitude from \( A \), so \( \angle FAS = \angle HAS = \angle FHK \).

Likewise, \( \angle FBS = \angle FKH \). Now \( S \), being the intersection of the perpendicular bisectors of \( FH \) and \( FK \), is the circumcenter of \( \triangle FHK \). Consider \( \angle FHK \) as an inscribed angle in the circumcircle of \( \triangle FHK \); then \( \angle FHK = \frac{1}{2} \angle FSK = \angle FSB \). Similarly \( \angle FBS = \angle FSA \). \( \Box \)

**Proof of Lambert's Theorem:** Let \( O \) be a point on the parabola between \( A \) and \( B \), \( PQ \) a tangent at \( O \) with \( P \) on \( AS \) and \( Q \) on \( BS \).
Apply the theorem on similar triangles to the parabola with tangents $AS, BS$ and with tangents $AP, QP$ to get $\angle FAS = \angle FSB$ and $\angle FAP = \angle FPO$ whence $\angle FSQ = \angle FPQ$. It follows from this that $P$ lies on the circumcircle of $\triangle FSQ$, i.e., $FPSQ$ is a cyclic quadrilateral.

**Construction** of the parabola from four tangents:

1. Choose two of the four tangent triangles determined by the four tangent lines.
2. Construct the circumcircle for both triangles.
3. The focus $F$ lies on their intersection.
4. Reflect $F$ about two of the tangents to get two points $H$ and $K$ on the directrix, and thus the directrix.
5. The parabola can then be "constructed" from the given focus and directrix.
Note 1. It’s a nice exercise to do the construction with some sort of geometry software.

Note 2. Since the point of intersection of two parabola tangents lies on a line parallel to the parabola’s axis, and passing through the midpoint of the chord joining the points of tangency (see No. 51), it follows that no three tangent lines are concurrent, and no two are parallel. The four lines chosen must satisfy these conditions.

Note 3. Lambert’s theorem leads directly to a solution of the problem: find the locus of the foci of all parabolas that are tangent to three given lines. The locus is the circumcircle of the triangle determined by the lines.