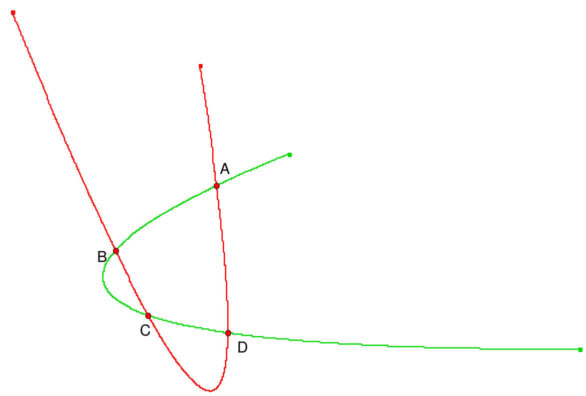
# 45. A Parabola from Four Points

**Draw a parabola that passes through four given points.**




This lovely problem was first solved by Newton in his celebrated *Philosophiae naturalis principia mathematica*, 1687, and then once again in 1707 in his *Arithmetica universalis*. The solution is based on the

**Auxiliary problem** Draw a parabola for which three points and the direction of the axis are known.

**The solution** of the auxiliary problem in turn is based on the following two theorems.

**Theorem I.** The centers of parallel chords of a parabola lie on a parallel to the axis.

**Theorem II.** The perpendicular bisector of a parabola chord and the perpendicular to the axis through the center of the chord cut off a segment on the axis of length equal to the distance of the focus to the directrix. (This distance is the length of the subnormal.)

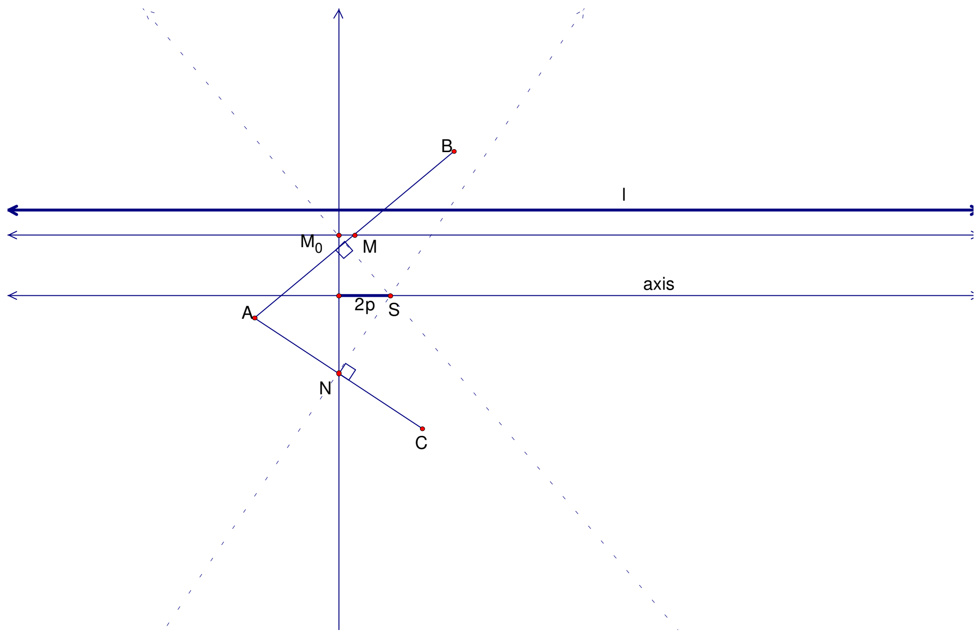
**Proof of I.** Let $y^2 = 4px$ be an equation of the parabola with focus $(p,0)$, directrix $x = -p$, and axis $y = 0$. If $A_1(x_1,y_1)$ and $A_2(x_2,y_2)$ are end points of a parabola chord, then $y_1^2 = 4px_1$ and $y_2^2 = 4px_2$. $y_2^2 - y_1^2 = 4p(x_2 - x_1)$, and the slope of $A_1A_2$ is $m = \frac{y_2-y_1}{x_2-x_1} = \frac{4p}{y_2+y_1}$ or $\frac{y_2+y_1}{2} = \frac{2p}{m}$. It follows that the centers of parallel chords lie on the line $y = \frac{2p}{m}$, which is parallel to the axis $y = 0$. □

**Proof of II.** The perpendicular bisector of chord $A_1A_2$ is $y - \frac{2p}{m} = \frac{-1}{m}\left(x - \frac{x_1+x_2}{2}\right)$ which has $x$-interecept $\left(\frac{x_1+x_2}{2} + 2p, 0\right)$. The perpendicular at the center of the chord meets the axis at $\left(\frac{x_1+x_2}{2}, 0\right)$ and the result follows. □

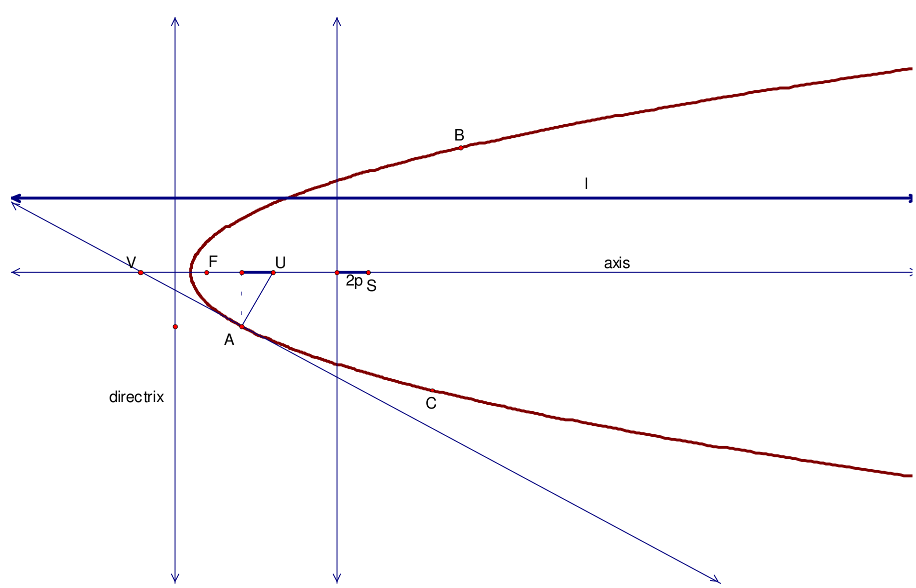
**Corollary of II.** If the midpoints of two parabola chords lie on a perpendicular to the axis, then the perpendicular bisectors of the chords instersect on the axis.

**Solution to the Auxiliary Problem** Let $A, B, C$ be the given points on the parabola, and line $\ell$ parallel to the axis. Let $M$ be the midpoint of $AB$ and $N$ be the midpoint of $AC$. Let the lines through $M \parallel \ell$ and through $N \perp \ell$ meet at $M_0$. By $I$, $M_0$ is the midpoint of a chord $A_0B_0 \parallel AB$. By the corollary of $II$, the perpendicular bisectors of $A_0B_0$ and

$AC$ intersect at a point $S$ on the axis at a distance of the subnormal $2p$ from the foot of the perpendicular of $N$ on the axis.

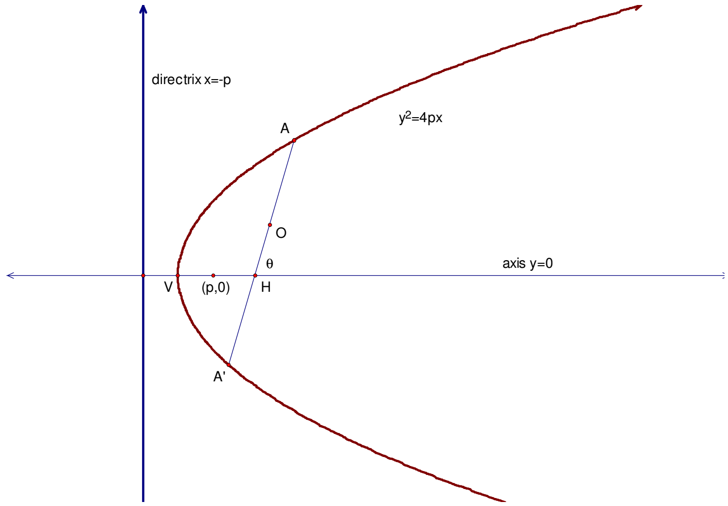



Making use of the subnormal $2p$ from $A$, draw the normal $AU$ and tangent $AV$ with $U$ and $V$ on the axis. The midpoint of $UV$ is the focus $F$; reflect $F$ about the tangent line to get a point on the directrix, and hence the directrix. This is sufficient to determine the parabola.




We need a further lemma to solve Newton's parabola problem.

**Lemma**. In the figure below, $O$ is any fixed point, $AA'$ any chord through $O$, and $H$ the intersection of chord $AA'$ and the axis.




Then $OA \cdot OA' \cdot \sin^2\theta$ is constant.

**Proof**. Let $O = (x_0, y_0)$ and $\rho = OA$. Then $A = (x_0 + \rho\cos\theta, y_0 + \rho\sin\theta)$ and from $y^2 = 4px$, it follows that $(\sin^2\theta)\rho^2 + 2(y_0\sin\theta - 2p\cos\theta)\rho + (y_0^2 - 4px_0) = 0$. The two solutions to this quadratic equation are $\rho = OA$ and $\rho' = OA'$. It follows that $\rho\rho' = \frac{y_0^2 - 4px_0}{\sin^2\theta}$. □

**Using** undirected distances, $OA \cdot OA' = \frac{|y_0^2 - 4px_0|}{\sin^2\theta} = \frac{|y_0^2 - 4px_0|}{y_0^2} OH^2$, and $\sqrt{OA \cdot OA'} = k \cdot OH$, where $k$ depends only on fixed point $O$ and the focus.

**With this lemma**, we can now obtain the following **solution to Newton's problem**: Let $A, B, C, D$ be the given points (no three on a line and not the vertices of a parallelogram). Draw the diagonals $AC$ and $BD$ of quadrilateral $ABCD$ and call their point of intersection $O$. Mark off on the diagonals from $OP = \sqrt{OA \cdot OC}$ and $OQ = \sqrt{OB \cdot OD}$. By the lemma, line $PQ$ is parallel to the parabola axis, and the problem now reduces to the auxiliary problem treated above. □

**Note 1**. There are generally two choices for $P$ and $Q$ above. This yields two possible axes, and two parabolas.

**Note 2**. It quadrilateral $ABCD$ is a parallelogram, we get a pair of parallel lines (or two pairs using both axes).

**Note 3**. It's a good excercise to do this "construction" with geometry software.

**Note 4**. Dörrie concludes with a solution using projective geometry.