67. Steiner's Division of Space by Planes

What is the maximum number of parts into which space can be divided by n planes?

This very interesting problem appears in Steiner's paper "Einige Gesetze über die Teilung der Ebene und des Raumes" (*Crelle's Journal,* Vol. 1, and Steiner's *Complete Works,* Vol. 1).

We start by solving the preliminary problem: **What is the maximum number of parts into which a plane can be divided by n lines**?

The number of parts will evidently be largest when no two lines are parallel, and no three lines are concurrent. We will assume that these two conditions are satisfied in the following argument, and denote the maximum number of plane regions by $\overline{n}$.

Thus, let the plane be divided into $\overline{n}$ regions by n lines. We now draw one additional line. This line is cut by each of the first n lines in n points (since no two lines are parallel and no three are concurrent), and the $n + 1$ intervals of the line determined by these points divides $n + 1$ of the $\overline{n}$ regions in two. It follows that

$$\overline{n+1} = \overline{n} + (n + 1).$$

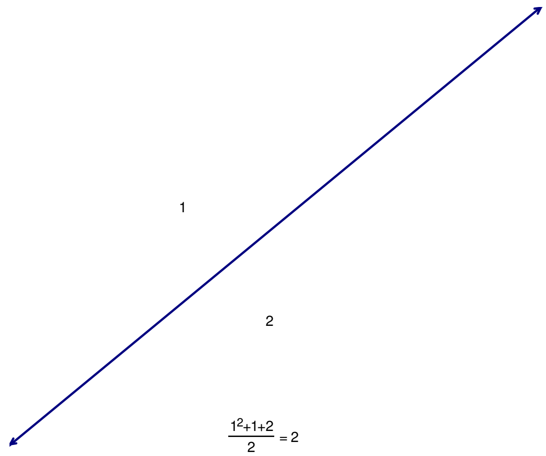
Apply this when $n = 0, 1, 2, \ldots$ to get

$$\begin{aligned} \overline{1} &= 1 + 1 \\ \overline{2} &= \overline{1} + 2 \\ \overline{3} &= \overline{2} + 3 \\ &\vdots \\ \overline{n} &= \overline{n-1} + n \end{aligned}$$

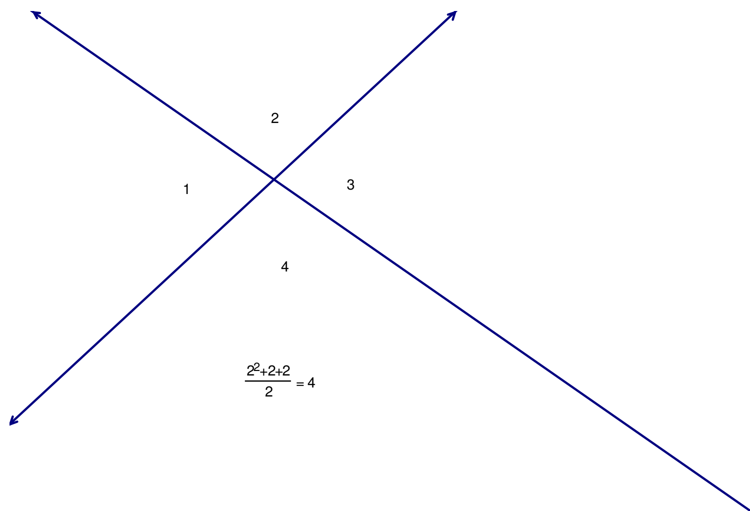
Addition of these equations results in

$$\overline{n} = 1 + (1 + 2 + 3 + \cdots + n) = 1 + \frac{n(n+1)}{2}. \tag{1}$$

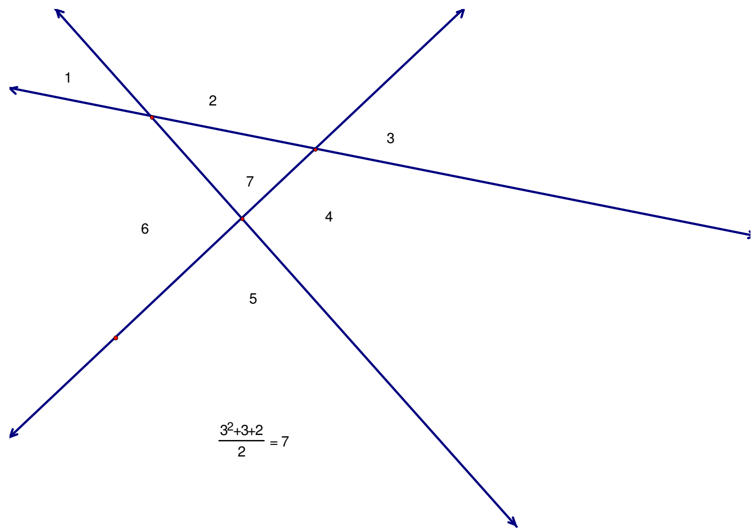
Thus the maximum number of parts into which a plane can be divided by n lines is $\frac{n^2+n+2}{2.}$. This is easy to verify for $n = 1, 2, 3$:



$$\frac{1^2+1+2}{2} = 2$$



$$\frac{2^2+2+2}{2} = 4$$



Now for the space problem! The number of spacial regions will be largest when no four planes intersect in a single point, and when the intersection of any three planes are non-parallel lines. We will assume that these two conditions are satisfied in the following argument, and denote the maximum number of spacial regions by $\tilde{n}$.

Thus, suppose that space is divided by n planes into $\tilde{n}$ regions. We now add an additional plane. By the conditions of the last paragraph, this plane is cut by the original n planes in n lines, no three of which are concurrent, and no two are parallel. The new $(n+1)^{\text{st}}$ plane is therefore divided by the n lines into $\overline{n}$ plane regions.

Each one of these $\overline{n}$ regions divides each spacial region it traverses in two, so that the addition of the $(n+1)^{\text{st}}$ plane increases $\tilde{n}$ by $\overline{n}$. Thus

$$\widetilde{n+1} = \tilde{n} + \overline{n}.$$

For $n = 1, 2, 3, \ldots$

$$\begin{aligned}
\tilde{1} &= 1 + 1 \\
\tilde{2} &= \tilde{1} + \overline{1} \\
\tilde{3} &= \tilde{2} + \overline{2} \\
&\vdots \\
\tilde{n} &= \widetilde{n-1} + \overline{n-1}
\end{aligned}$$

and addition yields the result

$$\tilde{n} = 2 + \overline{1} + \overline{2} + \overline{3} + \cdots + \overline{n-1}.$$

By (1),

$$
\begin{aligned}
\widetilde{n} &= 2 + \left(1 + \tfrac{1\cdot 2}{2}\right) + \left(1 + \tfrac{2\cdot 3}{2}\right) + \cdots + \left(1 + \tfrac{(n-1)n}{2}\right) \\
&= n + 1 + \tfrac{1}{2}(1 \cdot 2 + 2 \cdot 3 + \cdots + (n-1)n) \\
&= n + 1 + \tfrac{1}{2}\left(1^2 + 1 + 2^2 + 2 + \cdots + (n-1)^2 + (n-1)\right) \\
&= n + 1 + \tfrac{1}{2}\left(\left[1^2 + 2^2 + \cdots + (n-1)^2\right] + [1 + 2 + \cdots + (n-1)]\right) \\
&= n + 1 + \tfrac{1}{2}\left(\tfrac{(n-1)n(2n-1)}{6} + \tfrac{(n-1)n}{2}\right) \\
&= \tfrac{n^3+5n+6}{6}
\end{aligned}
$$

Thus the maximum number of parts into which space can be divided by n planes is

$$\frac{n^3 + 5n + 6}{6}.$$