69. The Shortest Distance Between Skew Lines

Find the angle and distance between two given skew lines. (Skew lines are non-parallel non-intersecting lines.)

This important problem is usually encountered in one of the following forms:

I. Find the angle and distance between two skew lines when a point on each line and the direction of each line are given - the former by coordinates and the latter by direction cosines.

II. Find the angle and distance between two opposite edges of a tetrahedron whose six edges are known.

The distance between two skew lines is naturally the shortest distance between the lines, i.e., the length of a perpendicular to both lines.

Solution of I. In the usual rectangular xyz-coordinate system, let the two points be $P_1 = (a_1, b_1, c_1)$ and $P_2 = (a_2, b_2, c_2); \overrightarrow{d} = P_1P_2 = \langle a_2 - a_1, b_2 - b_1, c_2 - c_1 \rangle$ is the direction vector from $P_1$ to $P_2$. Let $\overrightarrow{u_1} = \langle l_1, m_1, n_1 \rangle$ and $\overrightarrow{u_2} = \langle l_2, m_2, n_2 \rangle$ be unit direction vectors for the given lines; then the components of $\overrightarrow{u_i}$ are the direction cosines for the lines. Let $\omega$ be the sought-for angle and $k$ the sought-for minimum distance between the two lines.

The solution to this problem becomes very simple with the introduction of the dot (or scalar) product $\overrightarrow{u_1} \cdot \overrightarrow{u_2}$ and the cross product $\overrightarrow{u_1} \times \overrightarrow{u_2}$. We have $\cos \omega = \left| \frac{\overrightarrow{u_1} \cdot \overrightarrow{u_2}}{|| \overrightarrow{u_1} || \cdot || \overrightarrow{u_2} ||} \right| = \left| \overrightarrow{u_1} \cdot \overrightarrow{u_2} \right| = l_1l_2 + m_1m_2 + n_1n_2$ from which $\omega$ can be found. (We can assume that $\omega$ is acute, thus the absolute values.) $\overrightarrow{u_1} \times \overrightarrow{u_2}$ is orthogonal (perpendicular) to both lines, so the absolute value of the (scalar) projection of $\overrightarrow{d}$ onto $\overrightarrow{u_1} \times \overrightarrow{u_2}$ gives $k$. 

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Recall that the vector projection of \( \vec{b} \) on \( \vec{a} \) is \( \frac{\vec{a} \cdot \vec{b}}{\vec{a} \cdot \vec{a}} \vec{a} \) and the scalar projection is \( \frac{1}{|\vec{a}|} \vec{a} \cdot \vec{b} \). Thus \( k = \frac{\vec{d} \cdot \vec{a}_1}{|\vec{d} \times \vec{a}_1|} = \frac{|\vec{d} \cdot \vec{a}_2|}{\sin \omega} \) or

\[
k = \frac{\det \begin{bmatrix} a_2 - a_1 & b_2 - b_1 & c_2 - c_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{bmatrix}}{|\vec{d} \times \vec{a}_1|} / \sin \omega.
\]

**Note 1.** Since skew lines are not parallel, \( \vec{u}_1 \times \vec{u}_2 \neq \vec{0} \).

**Note 2.** Let \( X_1 = (x_1, y_1, z_1) \) and \( X_2 = (x_2, y_2, z_2) \) be closest points on the lines, and let \( \vec{k} = X_1X_2 \). Let \( r_i \) be the unique numbers such that \( X_i = P_i + r_i\vec{a}_i \). Since \( X_1X_2 = X_1P_1 + P_1P_2 + P_2X_2 \), we get \( \vec{k} = -r_1\vec{u}_1 + \vec{d} + r_2\vec{u}_2 \). Since \( \vec{k} \) is orthogonal to both lines, taking the dot product with \( \vec{u}_1 \) and \( \vec{u}_2 \) yields the system of linear equations:

\[
\begin{align*}
\vec{u}_1 \cdot \vec{u}_1 r_1 - \vec{u}_1 \cdot \vec{u}_2 r_2 - \vec{u}_1 \cdot \vec{d} &= 0 \\
\vec{u}_1 \cdot \vec{u}_2 r_1 - \vec{u}_2 \cdot \vec{u}_2 r_2 - \vec{u}_1 \cdot \vec{d} &= 0
\end{align*}
\]

in \( r_1 \) and \( r_2 \). There is a solution if \( \vec{u}_1 \cdot \vec{u}_2 \neq \pm 1 \), and this is the case since the lines are not parallel. Then \( X_i \) can be found.

**Solution of II.** Let the vertices of the tetrahedron be \( A, B, C, O \), the six edges \( BC, CA, AB, OA, OB, OC \) have lengths \( a, b, c, p, q, r \) respectively, and the vectors \( \vec{BC}, \vec{CA}, \vec{AB}, \vec{OA}, \vec{OB}, \vec{OC} \) be \( \vec{d}, \vec{c}, \vec{b}, \vec{p}, \vec{q}, \vec{r} \) respectively.
Let the angle and distance between the two opposite edges $\vec{c}$ and $\vec{r}$ be $\omega$ and $k$ respectively.

**Determination of $\omega$.** First of all,

$$\vec{c} + \vec{r} = \overrightarrow{AB} + \overrightarrow{OC}$$

$$= \overrightarrow{AO} + \overrightarrow{OB} + \overrightarrow{OA} + \overrightarrow{AC}$$

$$= \overrightarrow{OB} + \overrightarrow{AC}$$

$$= \vec{q} - \vec{b},$$

and thus

$$(\vec{c} + \vec{r}) \cdot (\vec{c} + \vec{r}) = (\vec{c} + \vec{r}) \cdot (\vec{q} - \vec{b}) = \vec{c} \cdot \vec{q} + \vec{q} \cdot \vec{r} - \vec{b} \cdot \vec{c} - \vec{b} \cdot \vec{r}.$$  

However,

$$(\vec{c} + \vec{r}) \cdot (\vec{c} + \vec{r}) = \vec{c} \cdot \vec{c} + \vec{r} \cdot \vec{r} + 2\vec{c} \cdot \vec{r} = c^2 + r^2 + 2cr \cos \omega$$

too, and

$$2\vec{c} \cdot \vec{q} = c^2 + q^2 - p^2, \quad 2\vec{q} \cdot \vec{r} = q^2 + r^2 - a^2, \quad 2\vec{b} \cdot \vec{c} = a^2 - b^2 - c^2, \quad 2\vec{b} \cdot \vec{r} = p^2 - b^2 - r^2.$$

Note that when the law of cosines is used with $\triangle ABC$, $a^2 = b^2 + c^2 - 2bc \cos \angle BAC$, and $\angle BAC$ and the angle $\theta$ between $\vec{b}$ and $\vec{c}$ are supplemenatal, so

$$a^2 = b^2 + c^2 + 2bc \cos \theta = b^2 + c^2 + 2\vec{b} \cdot \vec{c}. \quad \text{(Similarly for $\triangle OAC$.)}$$

It follows that
\[ 2cr \cos \omega = (\vec{c} + \vec{r}) \cdot (\vec{c} + \vec{r}) - c^2 - r^2 \]
\[ = \vec{c} \cdot \vec{c} + \vec{q} \cdot \vec{q} - \vec{b} \cdot \vec{b} - \vec{r} \cdot \vec{r} - c^2 - r^2 \]
\[ = a^2 + b^2 + c^2 - p^2 + q^2 + r^2 - c^2 - r^2 \]
\[ = b^2 + q^2 - a^2 - p^2 \]

and \( \omega \) can be found. (We can always choose \( \omega \) in the range 0° to 90°, since when two lines intersect, one of the vertical angle pairs is in this range, the other being supplemental. Note that \( b \) and \( q \), and \( a \) and \( p \) are lengths of opposite sides of the tetrahedron.)

Calculation of \( k \). Let the volume of the tetrahedron be \( T \). By No. 68, we can consider this quantity known. Translate vector \( \vec{r} \) parallel to itself so that its starting point (initial point or tail) is at \( A \); call the translated end point (or head) \( Q \). Then \( AQ \parallel OC \).

\( \triangle CQA \approx \triangle AOC \) (SSS), and thus tetrahedrons \( CQAB \) and \( AOCB \) have the same volume \( T \). Now consider \( \triangle QAB \) as the base of tetrahedron \( CQAB \) and \( C \) as its apex. The base area is \( \frac{1}{2} \cdot AQ \cdot AB \cdot \sin \angle QAB = \frac{1}{2} \cdot rc \sin \omega \). (Remember that \( \omega \) is the angle between \( \vec{c} \) and \( \vec{r} \).) To find the altitude of \( CQAB \) as the (perpendicular) distance from \( C \) to the plane \( QAB \), note that the plane \( QAB \) can be generated by translating line \( OC \) parallel to itself along line \( AB \). Then line \( OC \) lies in a plane parallel to plane \( QAB \). It follows that the altitude from \( C \) to \( QAB \) is \( k \), the shortest (perpendicular) distance between lines \( OC \) and \( AB \) (and between the two planes). The volume of tetrahedron \( CQAB \) is then \( \frac{1}{3} \cdot \frac{1}{2} \cdot rc \sin \omega \cdot k \), and thus \( 6T = kcr \sin \omega \) or

\[ k = \frac{6T}{cr \sin \omega}. \]

Corollary. \( |\vec{c} \times \vec{r}| = |\vec{c}| |\vec{r}| \sin \omega = cr \sin \omega \). With \( \vec{k} \) denoting the shortest vector between
lines $OC$ and $AB$, we have $6T = \left| \vec{k} \cdot \vec{c} \times \vec{r} \right|$. This is sometimes expressed as the

**Theorem.** The mixed product of the two opposite sides of a tetrahedron and the distance between them (all thought of as vectors) equals six times the volume of the tetrahedron.

A direct consequence of this is the famous

**Theorem of Steiner.** All tetrahedrons having two opposite edges of given length lying on two fixed lines have the same volume.