The Nilpotent Filtration in Group Cohomology

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Abstract

Let $P$ be a finite $p$-group and let $e$ be an idempotent in $\mathbb{F}_p[\text{Out}(P)]$. In this dissertation we explore the Krull dimension of $eH^*(P; \mathbb{F}_p)$. It is known that this dimension cannot exceed the largest rank of an elementary abelian $p$-subgroup of $P$. We investigate conditions on $P$ which ensure that $\dim(eH^*(P))$ is maximal.

The nilpotent filtration of the category of unstable modules over the Steenrod algebra plays a key role in the solutions we present. In particular, the dimension of a module depends only on the size of the subquotients in its nilpotent filtration. We also rely on the descriptions of the localization of $\mathcal{U}$ with respect to the categories $\mathcal{Nil}_n$ given by H. W. Henn, J. Lannes, and L. Schwartz in [19] and [20].

Our main results come in the form of two separate group theoretic criteria. For a group $P$, $\dim(eH^*(P))$ is maximal if:

- $P$ has an elementary abelian $p$-subgroup of maximal rank which is both normal in $P$ and self-centralizing; or

- all elements of order $p$ are central.
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“Here I raise my Ebenezer,
Hither by Thy help I’m come.”

Soli Deo Gloria