Introduction

Let P be a finite p-group and let $H^*(P)$ denote $H^*(P;\mathbb{F}_p)$, the mod p-cohomology of P. It is well known that $H^*(P)$ is a module over $\mathbb{F}_p[\mathrm{Out}(P)]$, where $\mathrm{Out}(P)$ is the outer automorphism group of P. If e is a nonzero idempotent element of $\mathbb{F}_p[\mathrm{Out}(P)]$, we are interested in the size of $eH^*(P)$ as a graded vector space over $\mathbb{F}_p$. In 1972, D. Quillen ([30]) showed that the Krull dimension of $H^*(P)$ is equal to the rank of the largest elementary abelian p-subgroup of P. We will denote this rank by $\mathrm{r}_p(P)$. In this dissertation we consider the following question.

Question 1 *Does the equality* $\dim(eH^*(P)) = \mathrm{r}_p(P)$ *always hold?*

Throughout our work we exploit the structure of $H^*(P)$ as an unstable module over the Steenrod algebra. The category of all such objects is denoted $\mathcal{U}$. For each $n \geq 1$, there is a notion of *n-nilpotence*, and the full subcategory of $\mathcal{U}$ consisting of n-nilpotent modules is denoted $\mathcal{N}il_n$. These are nested subcategories, forming a decreasing filtration of $\mathcal{U}$,

$$\cdots \subseteq \mathcal{N}il_3 \subseteq \mathcal{N}il_2 \subseteq \mathcal{N}il_1 = \mathcal{N}il \subseteq \mathcal{U},$$

known as the *nilpotent filtration*. If $\operatorname{nil}_n M$ denotes the largest submodule of M contained in $\mathcal{N}il_n$, we also have a filtration of each unstable module M:

$$\cdots \subseteq \operatorname{nil}_3 M \subseteq \operatorname{nil}_2 M \subseteq \operatorname{nil}_1 M \subseteq M.$$

The subquotients in this filtration have a particularly nice description. An unstable module is called *reduced* if it contains no nontrivial suspensions. For each $s \geq 0$, we are able to define a reduced module $R_s(M)$ by the formula

$$\Sigma^s R_s(M) \cong \operatorname{nil}_s M / \operatorname{nil}_{s+1} M.$$

The existence of the modules $R_s(M)$ is one of the defining characteristics of the nilpotent filtration, and we will use these modules to gather information about M.

In order to calculate the dimension of an unstable module, it will be necessary to focus on the subcategory of $\mathcal{U}$ denoted $K_{fg} - \mathcal{U}$. If M is a module in $K_{fg} - \mathcal{U}$, the nilpotent filtration of M is finite, meaning that there are a finite number of reduced modules $R_s(M)$.

Every unstable reduced module embeds in a terminal reduced module called its $\mathcal{N}il$-*closure*. Our approach to Question 1 involves drawing conclusions about the dimension of a module M from the dimension of the $\mathcal{N}il$-closure of the modules $R_s(M)$. We denote the $\mathcal{N}il$-closure of $R_s(M)$ by $\overline{R}_s(M)$. If M is a module in $K_{fg} - \mathcal{U}$, then $\dim(\overline{R}_s(M))$ is defined for every s, and we prove the following theorem in Section 3.4. We will apply this theorem when $M = eH^*(P)$.

Theorem *If* $M \in K_{fg} - \mathcal{U}$, *then* $\dim(M) = \max\{\dim(\overline{R}_s(M))\}$.

The category $\mathcal{A}(P)$ consists of the elementary abelian p-subgroups of P. Our first result only requires the use of $\overline{R}_0(H^*(P))$ in the above theorem, which is familiar as Quillen's inverse limit $\varprojlim_{\mathcal{A}(P)} H^*(V)$.

H. W. Henn, J. Lannes, and L. Schwartz studied the localization of $\mathcal{U}$ with respect to $\mathcal{N}il$ in [19], and they showed that $\mathcal{U}/\mathcal{N}il$ is equivalent to a certain category of functors. Let W denote an elementary abelian p-group and consider the set $\mathrm{Hom}(W, P)$. Via conjugation, there is an action of P on this set, and we denote the set of orbits of this action by $\mathrm{Rep}(W, P)$. In the equivalence of Henn, Lannes, and Schwartz, the module $\overline{R}_0(H^*(P))$ corresponds to the functor that sends W to $\mathbb{F}_p^{\mathrm{Rep}(W,P)}$.

It is easy to see that $\mathrm{Rep}(W, P)$ is an $\mathrm{Out}(P)$-set: if $[\alpha] \in \mathrm{Rep}(W, P)$ and $[f] \in \mathrm{Out}(P)$, then $[f] \cdot [\alpha] := [f \circ \alpha]$. If V is an elementary abelian p-subgroup of P, and ϕ is a map in $\mathrm{Hom}(W, P)$ with $\phi(W) = V$, we define $\mathrm{Out}(P)_V$ by the formula

$$\mathrm{Out}(P)_V := \{[f] \in \mathrm{Out}(P) \mid [f] \cdot [\phi] = [\phi]\}.$$

(This is independent of the choice of W and ϕ.) If $\mathrm{Out}(P)_V$ is a p-group, then it is easy to check that every simple $\mathbb{F}_p[\mathrm{Out}(P)]$-module occurs as a composition factor in $\mathbb{F}_p[\mathrm{Out}(P)/\mathrm{Out}(P)_V]$. In turn, we show that this implies $\dim(eH^*(P)) \geq \mathrm{rk}(V)$ for every nonzero idempotent e. This leads to the following result, in which we give group theoretic conditions which ensure that $\mathrm{Out}(P)_V$ is a p-group. (This is a slightly abbreviated version of Corollary 4.19.)

Theorem *If P contains an elementary abelian p-subgroup of maximal rank that is self-centralizing and normal, then* $\dim(eH^*(P)) = \mathrm{r}_p(P)$.

In Chapter 5 we turn to a description of the modules $\overline{R}_n(H^*(P))$, and for this we require a second paper of Henn, Lannes, and Schwartz. In [20] these authors examined localization of $\mathcal{U}$ away from $\mathcal{N}il_n$ for all $n \geq 1$. If $L_n : \mathcal{U} \to \mathcal{U}$ is the localization functor corresponding to $\mathcal{N}il_n$, then according to [20, Theorem I.5.5] one can calculate $L_n(H^*(P))$ by the following limit over $\mathcal{A}(P)_\sharp$:

$$\lim_{E \xrightarrow{\alpha} E'} \left[\mathrm{Eq} : \left\{ H^*(E) \otimes \Big(H^*(C_P(E')) \Big)^{<n} \overset{\mu(\alpha)}{\underset{\nu(\alpha)}{\rightrightarrows}} H^*(E) \otimes \Big(H^*(E \times C_P(E')) \Big)^{<n} \right\} \right]. \quad (*)$$

Let C denote the largest central elementary abelian p-subgroup of P, and let $\mathcal{A}_C(P)$ denote the full subcategory of $\mathcal{A}(P)$ whose objects are those in $\mathcal{A}(P)$ which contain C. In Chapter 6 we prove that there is an easier way to calculate $L_n(H^*(P))$.

Theorem *The module* $L_n(H^*(P))$ *can be calculated by the formula in* $(*)$ *as a limit over* $\mathcal{A}_C(P)_\sharp$ *instead of* $\mathcal{A}(P)_\sharp$.

After we prove that the calculation of $L_n(H^*(P))$ can be taken over $\mathcal{A}_C(P)_\sharp$ instead of $\mathcal{A}(P)_\sharp$, we show that the same holds for $\overline{R}_n(H^*(P))$. For certain groups, this reduction in the formula for $\overline{R}_n(H^*(P))$ provides a second answer to Question 1. If a group G has the property that every element of order p is central, we say that G is a *p-central group*. In this situation, G has a unique maximal elementary abelian p-subgroup. Using this structure, we prove the following theorem, which appears later as Theorem 7.4.

Theorem *If* P *is a p-central group, then* $\dim(eH^*(P)) = \mathrm{r}_p(P)$.

This theorem was essentially proved by J. Martino and S. Priddy in [26]. In this paper the authors posed a version of Question 1; we will explain the connections between [26] and the question we consider in Section 1.4.

In addition to the calculations we have described thus far, we include several examples in Chapter 8. Since p-central groups can be viewed as central extensions, we will use the Lyndon-Hochshield-Serre spectral sequence to calculate the cohomology of a p-central group of order 32. Additionally, we present explicit descriptions of the modules $R_n(H^*(P))$ and $\overline{R}_n(H^*(P))$ in the cases of the quaternion group and the group 32#18. Finally, since calculations of the mod 2-cohomology of a large number of 2-groups are readily available, one can check the progress toward answering Question 1 for 2-groups. A careful inspection reveals only a single 2-group of order dividing 64 for which Question 1 is not settled. We include a table at the end of Chapter 8 containing this information.